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On characters and formal degrees of discrete series of affine Hecke algebras of classical types

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Abstract

We address two fundamental questions in the representation theory of affine Hecke algebras of classical types. One is an inductive algorithm to compute characters of tempered modules, and the other is the determination of the constants in the formal degrees of discrete series (in the form conjectured by Reeder [26]). The former is completely different from the Lusztig-Shoji algorithm [27, 16], and it is more effective in a number of cases. The main idea in our proof is to introduce a new family of representations which behave like tempered modules, but for which it is easier to analyze the effect of parameter specializations. Our proof also requires a comparison of the C^*-theoretic results of Opdam, Delorme, Slooten, Solleveld [23, 9, 30, 24, 25], and the geometric construction from [12, 13, 7].

1 Introduction

In this paper, we consider two basic questions in the study of affine Hecke algebra of classical types with unequal parameters. The first one is the characters of tempered modules. The classical approach (for W-characters) is via the Lusztig-Shoji algorithm ([27, 16]), which computes the generalized Green functions in terms of geometric data. We present an alternative approach, namely an inductive algorithm on the rank and the ratio of parameters of the affine Hecke algebra. Since the Lusztig-Shoji algorithm treats each (geometric) ratio of parameters individually, our algorithm has some advantage, particularly if one is interested in the connection between two different ratios. As a consequence of this new algorithm, we answer the second basic question, namely the determination of the rational constants in the formal degrees of discrete series. Our result confirms the expected values of these constants, motivated by the study of complex smooth representations of p-adic groups (see the discussion after Theorem C). More generally, in conjunction with Bushnell-Henniart-Kutzko [4] Theorem B, this provides an explicit formula for formal degrees of discrete series of p-adic groups of classical types for many (if not all) Bernstein blocks.

To explain our results more precisely, let $\mathbb{H}_n(q,u,v)$ be the affine Hecke algebra of type C_n with parameters q,u,v (see §2.2). We specialize to the cases $\mathbb{H}_{n,m} = \mathbb{H}_n(q,q^m,q^m)$ and

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$\mathbb{H}'_{n,m} = \mathbb{H}_n(q, q^{2m}, 1)$, with $m \in \mathbb{R}$. These are the affine Hecke algebras with two parameters of type C_n and (up to central extension) of type B_n, respectively. Let W_n denote the Weyl group of type BC_n, and let $\widehat{W_n}$ denote the set of irreducible W_n-representations. In order to explain our results on the W_n-character of tempered modules, we also restrict to the so-called positive real central character case (see section §2.3).

There is a correspondence between the set of discrete series with real central characters of $\mathbb{H}_{n,m}$ and $\mathbb{H}'_{n,m}$. For every partition σ of n, there is a real central character c^σ_m attached to σ and m. When m is generic, i.e., $m \notin \frac{1}{2}\mathbb{Z}$, there exists a unique discrete series with central character c^σ_m, and moreover, every discrete series module of $\mathbb{H}_{n,m}$ or $\mathbb{H}'_{n,m}$ has central character c^σ_m for some partition σ of n (Opdam [23]). Therefore, we can regard the discrete series with real central character as belonging to families $\{\mathsf{ds}_m(\sigma)\}_m$, $\{\mathsf{ds}'_m(\sigma)\}_m$ for $\mathbb{H}_{n,m}$ and $\mathbb{H}'_{n,m}$, respectively, indexed by partitions σ of n. Then, we have

$$\mathsf{ds}_m(\sigma) \cong \mathsf{ds}'_m(\sigma), \text{ as } W_n\text{-modules.} \tag{1.1}$$

If m_0 is a critical parameter, i.e., $m_0 \in \frac{1}{2}\mathbb{Z}$, then it is known by Opdam-Solleveld [25] that every discrete series of $\mathbb{H}_{n,m_0}$ (resp. $\mathbb{H}'_{n,m_0}$) is obtained as a limit $m \to m_0$ of certain $\mathsf{ds}_m(\sigma)$ (resp. $\mathsf{ds}'_m(\sigma)$). Here, $\lim_{m\to m_0}$ is in the sense of [7] §2.4.

As already mentioned, we are interested in the W_n-character of $\mathsf{ds}_m(\sigma)$. Our strategy is as follows: for $m > n-1$, the $\mathsf{ds}_m(\sigma)$ is simple and does not change as a W_n-module by [7]. Namely, we have

$$\mathsf{ds}_m(\sigma) \mid_{W_n} = \{\emptyset, {}^{\mathsf{t}}\sigma\}, \text{ if } m > n-1, \tag{1.2}$$

where the notation for $\widehat{W_n}$ is via bipartitions (c.f. §3.5). Then we keep track how the W_n-character of $\mathsf{ds}_m(\sigma)$ changes as m varies towards $-\infty$. It can only change when m passes through a critical value m_0, but this is a subtle problem. We resolve this difficulty by considering a larger family of irreducible modules $\mathcal{D}_{m_0}(\sigma)$ depending on $m \neq m_0$, $m_0 - \frac{1}{2} < m < m_0 + \frac{1}{2}$ with the following properties:

(a) π has central character c^σ_m (same as $\mathsf{ds}_m(\sigma)$); and

(b) $\lim_{m\to m_0} \pi$ is tempered.

For lack of a better name, we call such modules delimits of tempered modules, or tempered delimits for short. For example, we have $\mathsf{ds}_m(\sigma) \in \mathcal{D}_{m_0}(\sigma)$ for both $m_0 - \frac{1}{2} < m < m_0$ and $m_0 < m < m_0 + \frac{1}{2}$. It should be noted that the modules appearing as $\lim_{m\to m_0} \pi$, $\pi \in \mathcal{D}_{m_0}(\sigma)$ can be thought of as analogues of the nondegenerate limits of discrete series in the sense of Knapp-Stein [15], §XIV.17 and Theorem 14.92.

A main technical achievement of this paper is the following:

Theorem A (Corollary 3.23). *Assume that $m_0 \in \frac{1}{2}\mathbb{Z}$. Then for every $\pi \in \mathcal{D}_{m_0}(\sigma)$, $\lim_{m\to m_0} \pi$ is an irreducible $\mathbb{H}_{n,m_0}$-module. In particular, $\lim_{m\to m_0} \mathsf{ds}_m(\sigma)$ is irreducible.*

This theorem, proved as a corollary of basic properties of tempered delimits (Theorems 3.15, 3.16), represents the basis for our algorithm. By the geometry of tempered delimits, we deduce:

Theorem B (Formula (3.17)). *For every $m_0 \in \frac{1}{2}\mathbb{Z}$, we have the following equality inside the Grothendieck group of $\mathbb{H}_{n,m_0}$-modules:*

$$[\lim_{m'\to m_0} \mathsf{ds}_{m'}(\sigma)] \pm [\lim_{m\to m_0} \mathsf{ds}_m(\sigma)] = \sum (\pm)[L^{\mathsf{A}} \circledast L'], \tag{1.3}$$

where the real variables m, m' satisfies $m_0 - \frac{1}{2} < m' < m_0 < m < m_0 + \frac{1}{2}$. Here $L^{\mathsf{A}} \circledast L'$ denotes parabolic induction from a tempered module L^{A} of an affine Hecke algebra of type A and a discrete series L' of a type C affine Hecke algebra. Moreover, all the terms in the right hand side are induced from proper Levi subalgebras.

We remark that the right hand side of (1.3) looks obscure here but the actual expression is explicit and precise (see (3.17) for details). Moreover, (1.3) implies certain relations between the W-characters of classical and exotic Springer fibers (Corollary 3.26).

In addition, if we assume, by induction on the rank of the Hecke algebra, that we know the discrete series character of smaller affine Hecke algebras of type C, then we easily deduce the character of the right hand side of (1.3). Hence, if we know the character of either $\mathsf{ds}_{m'}(\sigma)$ or $\mathsf{ds}_m(\sigma)$, then we deduce the other. Thanks to (1.2), we always know the W_n-character of $\mathsf{ds}_m(\sigma)$ for $m \gg 0$. Our algorithm 3.30 (on the W_n-characters of tempered delimits) is an implementation of these observations. As we see in Remark 3.31, our construction also gives an inductive algorithm to compute weight characters of tempered delimits with respect to the abelian subalgebra that appears in the Bernstein-Lusztig presentation ([20] §3).

In section 4, we use the W-character algorithm to complete the computation of the formal degree for the affine Hecke algebra $\mathbb{H}_n(q, q^{m_+}, q^{m_-})$ of type C_n, where $q > 1$ and $m_\pm \in \mathbb{R}$. All affine Hecke algebras of classical types are (up to central extensions) particular cases of this one. Denote the roots of type C_n by R_n, and let R_n^{sh} and R_n^{lo} denote the short and long roots, respectively. From [25], the formal degree of a discrete series π with central character s (not necessarily positive real) is known to equal

$$\mathrm{fd}(\pi) = \frac{C_\pi \; q^{n^2-n} q^{nm_+} \prod'_{\alpha \in R_n} (\alpha(s) - 1)}{\prod'_{\alpha \in R_n^{\mathsf{sh}}} (q\alpha(s) - 1) \prod'_{\alpha \in R_n^{\mathsf{lo}}} (q^{\frac{m_+ + m_-}{2}} \alpha(s)^{1/2} - 1) \prod'_{\alpha \in R_n^{\mathsf{lo}}} (q^{\frac{m_+ - m_-}{2}} \alpha(s)^{1/2} + 1)}, \tag{1.4}$$

where $\prod'$ means that the product is taken only over the nonzero factors. From Opdam-Solleveld [24], it is known that the constants C_π are rational numbers, and the question is to determine them explicitly. We use an Euler-Poincaré formula which expresses the formal degree as an alternating sum depending on the W-character of the discrete series (see 4.3) as in Reeder [26]. This formula itself is proved in Schneider-Stuhler [29] (for p-adic groups) and in Opdam-Solleveld [24] (for affine Hecke algebras).

Following [12], we say that (m_+, m_-) are generic if $|m_+ \pm m_-| \notin \{0, 1, 2, \ldots, 2n-1\}$. We use Theorem B to find that the constants C_π for generic (m_+, m_-) do not depend (up to sign) on m. Combined with an explicit calculation in an asymptotic region of the parameters (m_+, m_-) and a certain limiting process, we obtain:

Theorem C (Theorem 4.7 and Corollary 4.8)**.** *Let π be a discrete series with arbitrary central character for the affine Hecke algebra $\mathbb{H}_n(q, q^{m_+}, q^{m_-})$, where $q > 1$ and $m_\pm \in \mathbb{R}$. Then, the constant in (1.4) is (up to sign) $C_\pi = \frac{1}{2}$.*

The scalar $1/2$ comes from $1/|\Omega|$, where Ω denotes the quotient of the character lattice by the root lattice for the Hecke algebra we consider. In §4.2 (4.15,4.17,4.19), we explain the implications of Theorem C for the affine Hecke algebras of types $\mathsf{C}_n, \mathsf{B}_n, \mathsf{D}_n$, respectively.

As mentioned previously, this calculation has consequences for p-adic groups as well. The expected stability of L-packets of discrete series for a p-adic group $\mathcal{G}$ implies that the formal degrees of discrete series in the same L-packet have to be proportional, with the proportionality constants being the multiplicities of discrete series in the stable L-packet sum. In the case discrete series are in the scope of the Deligne-Langlands-Lusztig correspondence [21], there is a precise conjecture for the values of the constants formulated in

[26] (0.5). Particularly, when the p-adic group is of classical type (other than the quasisplit triality form of D_4), those discrete series are controlled by that of various affine Hecke algebras of classical types. For example, with our notation, the Iwahori cases for split p-adic classical adjoint groups $SO(2n+1)$, $PSp(2n)$, $PSO(2n)$ correspond to the Hecke algebras $\mathbb{H}'_{n,\frac{1}{2}}$, $\mathbb{H}_{n,1}$ and $\mathbb{H}'_{n,0}$, respectively. (In fact the last algebra is central extension of the Iwahori-Hecke algebra for type D_n, but for our purposes, this is sufficient; see Proposition 3.34). Using the correspondence between the Plancherel formula for groups and for the Hecke algebras ([4]), and taking also into account Hiraga-Ichino-Ikeda [11] §3.4, one verifies that the values of the constants obtained from Theorem C match the expected values from p-adic groups.

The organization of the paper is as follows. In §2 we recall the geometric setup, and we fix the notation for the affine Hecke algebras. Then we study a number of properties of Langlands quotients of parabolically induced modules which we need in §3. In §3, we define and classify the tempered delimits, and prove the results about irreducibility under deformations in the parameter m. We present the inductive algorithm for the W-characters of discrete series and tempered modules. In §4, we calculate the constants in formal degrees.

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Convention

For two sets $J_1, J_2 \subset \mathbb{Z}$, we define $J_1 < J_2$ if and only if $j_1 < j_2$ for every $j_1 \in J_1$ and $j_2 \in J_2$.

Fix $\vec{q} = (q_1, q) \in \mathbb{R}^2$ so that $q > 1$ and $q_1 = q^m$ for some $m \in \mathbb{R}$. We say m is *generic* if and only if $m \notin \frac{1}{2}\mathbb{Z}$. A q-segment (or just a segment if there can be no possible confusion) is a sequence of positive real numbers of the form

$$a, aq, aq^2, \cdots, aq^M \text{ for some } M \in \mathbb{Z}_{\geq 0}.$$

For two q-segments I_1, I_2, we define

$$\begin{aligned}
\mathtt{E}_m(I_1) :=& \prod_{a \in I_1} a, \quad e^+_m(I_1) = e_+(I_1) := \max I_1, \quad e^-_m(I_1) = e_-(I_1) := \min I_1,\\
I_1 \Subset I_2 & \Leftrightarrow \min I_2 < \min I_1 \text{ and } \max I_1 < \max I_2,\\
I_1 \lhd I_2 & \Leftrightarrow \min I_1 < \min I_2 \leq q \max I_1 < q \max I_2, \text{ and}\\
I_1 \unlhd I_2 & \Leftrightarrow \min I_1 \leq \min I_2 \leq q \max I_1 < q \max I_2\\
& \text{or } \min I_1 < \min I_2 \leq q \max I_1 \leq q \max I_2.
\end{aligned}$$

Finite collections of q-segments (with possible repetitions) are called q-multisegments (or just multisegments). The set of q-multisegments is denoted by $\mathsf{Q}(q)$. For $\mathbf{I}, \mathbf{I}' \in \mathsf{Q}(q)$, we write $\mathbf{I} \subset \mathbf{I}'$ if each segment of $\mathbf{I}$ gives a segment of $\mathbf{I}'$ with multiplicity counted.

For a partition λ, we set $|\lambda| := \sum_i \lambda_i$, $\lambda_i^< := \sum_{j<i} \lambda_j$, and $\lambda_i^{\leq} := \sum_{j \leq i} \lambda_j$. We denote by ${}^{\mathtt{t}}\lambda$ the transpose partition of λ.

For an algebraic variety $\mathcal{X}$ over $\mathbb{C}$, we denote by $H_\bullet(\mathcal{X})$ the total Borel-Moore homology with coefficients in $\mathbb{C}$.

2 Preliminaries

2.1 Basic geometric setup

We denote by $G_n = Sp(2n, \mathbb{C})$ the symplectic group with its maximal torus T_n and a Borel subgroup $B_n \supset T_n$. Let $R_n \supset R_n^+$ be the root systems of (G_n, T_n) and (B_n, T_n), respectively. We define $X^*(T_n)$ to be the character lattice of T_n with its natural orthonormal basis $\epsilon_1, \ldots, \epsilon_n$ so that

$$\begin{aligned} R_n^+ &= \{(\epsilon_i \pm \epsilon_j), i < j, 2\epsilon_i\} \subset R_n = \{\pm(\epsilon_i \pm \epsilon_j), i < j, \pm 2\epsilon_i\} \\ \check{R}_n^+ &= \{(\epsilon_i \pm \epsilon_j), i < j, \epsilon_i\} \subset \check{R}_n = \{\pm(\epsilon_i \pm \epsilon_j), i < j, \pm\epsilon_i\}, \end{aligned}$$

where $\check{R}_n \supset \check{R}_n^+$ is the dual root system. Let $\check{\alpha} \in \check{R}_n$ denote the coroot of $\alpha \in R_n$. Let $W_n := N_{G_n}(T_n)/T_n$ be the Weyl group of G_n. Let $V_n^{(1)} = \mathbb{C}^{2n}$ be the vector representation of G_n and let $V_n^{(2)} := \wedge^2 \mathbb{C}^{2n}$ be its second wedge. We define $\mathbb{V}_n := V_n^{(1)} \oplus V_n^{(2)}$ to be the 1-exotic representation of G_n. Let $\mathbb{V}_n^+$ be the sum of T_n-weight spaces of $\mathbb{V}_n$ for which the corresponding weights are in $\check{R}_n^+$. We define $W_n(s) := \{w \in W_n \mid \mathrm{Ad}(w)s = s\}$ for each $s \in T_n$. For $w \in W_n$, we fix a lift $\dot{w}$ of w in $N_G(T)$. We set $F_n := G_n \times^{B_n} \mathbb{V}_n^+$. We form a map

$$\mu_n : F_n = G_n \times^{B_n} \mathbb{V}_n^+ \longrightarrow \mathbb{V}_n \tag{2.1}$$

obtained as the anti-diagonal free B_n-quotient of the action map $G_n \times \mathbb{V}_n^+ \to \mathbb{V}_n$. For every semisimple element $a = (s, \vec{q})$, denote by F_n^a, $\mathbb{V}_n^a$, and μ_n^a, the a-fixed points and the restriction to the a-fixed points of $F_n, \mathbb{V}_n$, and μ_n, respectively. Moreover, for every subvariety Y we denote by Y^a, the intersection of Y with the a-fixed points.

We might drop the subscript n when the meaning is clear from the context.

2.2 Hecke algebras

We consider the affine Hecke algebras $\mathbb{H}'_n(q, u)$, $\mathbb{H}_n(q, u)$, and $\mathbb{H}''_n(q)$ of type B_n, C_n, and D_n, respectively, with positive real parameters u, v, according to the affine Coxeter diagrams

$$\widetilde{\mathsf{B}}_n : \quad \begin{tikzcd}[arrows=dash] q \arrow[r] & q \arrow[r] \arrow[d] & q \arrow[r] & \cdots \arrow[r] & q \arrow[r] & q \arrow[r, equal] & u \\ & q & & & & & \end{tikzcd}$$

$$\widetilde{\mathsf{C}}_n : \quad \begin{tikzcd}[arrows=dash] u \arrow[r, equal] & q \arrow[r] & q \arrow[r] & \cdots \arrow[r] & q \arrow[r] & q \arrow[r, equal] & u, \end{tikzcd} \text{ and}$$

$$\widetilde{\mathsf{D}}_n : \quad \begin{tikzcd}[arrows=dash] q \arrow[r] & q \arrow[r] \arrow[d] & q \arrow[r] & \cdots \arrow[r] & q \arrow[r] & q \arrow[r] \arrow[d] & q \, . \\ & q & & & & q & \end{tikzcd}$$

We consider them as subalgebras of certain specializations (see below) of the affine Hecke algebra $\mathbb{H}_n(q, u, v)$ of type C_n

$$\widetilde{\Pi}_n : \quad \begin{tikzcd}[arrows=dash] v \arrow[r, equal] & q \arrow[r] & q \arrow[r] & \cdots \arrow[r] & q \arrow[r] & q \arrow[r, equal] & u \end{tikzcd}$$

defined as a $\mathbb{C}$-algebra with the set of generators $N_0, N_1, \ldots, N_n$ subject to the relations:

- $(N_0 + 1)(N_0 - v) = 0 = (N_n + 1)(N_n - u)$ and $(N_i + 1)(N_i - q) = 0$ for $1 \leq i < n$;

- $N_iN_j = N_jN_i$ for $i - j \geq 2$, $N_iN_{i+1}N_i = N_{i+1}N_iN_{i+1}$ for $1 \leq i < n-1$;
- $(N_0N_1)^2 = (N_1N_0)^2$ and $(N_{n-1}N_n)^2 = (N_nN_{n-1})^2$.

Let $\mathbb{H}_n^{\mathsf{A}}$ be the affine Hecke algebras of type $GL(n)$ with parameter q, which can be realized as a subalgebra of $\mathbb{H}_n(q,u,v)$ generated by $N_1, \ldots, N_{n-1}$, and $N_1N_2 \cdots N_{n-1}N_nN_{n-1} \cdots N_1N_0^{-1}$.

We remark that $\mathbb{H}_n(q,u)$ is obtained from $\mathbb{H}_n(q,u,v)$ by making the specialization $u = v$. Also, a central extension $\mathbb{H}_n^{\mathsf{B}}(q,u)$ of $\mathbb{H}'_n(q,u)$ is obtained from $\mathbb{H}_n(q,u,v)$ by making the specialization $v = 1$.

We define an algebra involution $\psi : \mathbb{H}_n(q,1,1) \to \mathbb{H}_n(q,1,1)$ as:

$$\psi(N_i) = N_i \text{ if } i \neq n, \text{ and } \psi(N_n) = -N_n.$$

A central extension $\mathbb{H}_n^{\mathsf{D}} = \mathbb{H}_n^{\mathsf{D}}(q)$ of $\mathbb{H}''_n(q)$ is realized as the ψ-invariant part of $\mathbb{H}_n(q,1,1)$ (see for example [25] §6.4).

We denote the finite Weyl groups of type BC_n and D_n by W_n and W_n^{D}, respectively. We denote the affine Weyl groups of type B_n and C_n by $\widetilde{W}_n$ and $\widetilde{W}'_n$, respectively.

We define $\mathbb{H}_{n,m} := \mathbb{H}_n(q,q^m)$, and $\mathbb{H}_{n,m}^{\mathsf{B}} := \mathbb{H}'_n(q,q^{2m})$. The representation theories of $\mathbb{H}_{n,m}$ and $\mathbb{H}_{n,m}^{\mathsf{B}}$ are known to be equivalent to that of $\mathbb{H}_n(q,u,v)$ with $u = q^{(m+m')}$ and $v = q^{(m-m')}$ for an arbitrary $m' \in \mathbb{R}$, once we fix a positive real central character (Lusztig [20, 22], see also [12] §3 and §2.3 below for the geometric explanation). Moreover, these equivalences preserve W_n-characters, and the notion of tempered modules and discrete series (see for example [22], §3). Since a central extension does not have an effect at the level of representations with positive real central character, we only deal with the representation theory of $\mathbb{H}_{n,m}$ and $\mathbb{H}_n^{\mathsf{D}}$ in this section and §3. In addition, we sometimes drop the subscript m for the sake of simplicity.

We also need in §4 the finite Hecke algebra of type BC_n with parameters q, u according to the Coxeter diagram

$$q \longrightarrow q \longrightarrow \cdots \longrightarrow q \longrightarrow q \Longrightarrow u$$

we denote it by $\mathbb{H}_n^f(q,u)$, or by $\mathbb{H}_{n,m}^f$ when $u = q^m$. We denote by $\mathbb{H}_n^{\mathsf{D},f}$ the ψ-invariant part of $\mathbb{H}_n^f(q,1)$. We remark that the irreducible modules of $\mathbb{H}_{n,m}^f$ and $\mathbb{H}_n^{\mathsf{D},f}$ are in one-to-one correspondence with $\widehat{W_n}$ and $\widehat{W_n^{\mathsf{D}}}$, respectively.

Let R be a ring. Let M be a R-module and let L be an irreducible R-module. Then, we denote the Jordan-Hölder multiplicity of L in M as R-modules by $[M : L]_R$. If $R = \mathbb{H}_{n,m}$, then we drop the subscript R for the sake of simplicity.

2.3 Representation-theoretic setup

A result of Bernstein and Lusztig says that the center of $\mathbb{H}_n$ is

$$Z(\mathbb{H}_n) = \mathbb{C}[e^{\lambda}; \lambda \in X^*(T_n)]^{W_n}, \tag{2.2}$$

so the central characters of $\mathbb{H}_n$ are parameterized by W_n-conjugacy classes of semisimple elements $s \in T_n$. An element (or a central character) $s \in T_n$ is said to be positive real if $\epsilon_i(s) > 0$ for $i = 1, \ldots, n$.

We denote by $\mathfrak{Mod}_{\vec{q}}^n$ the category of finite-dimensional $\mathbb{H}_n$-modules with positive real central character. We set $\mathfrak{Mod}_{\vec{q}} := \bigcup_{n \geq 1} \mathfrak{Mod}_{\vec{q}}^n$. For a group H and $h \in H$, we denote by $R(H)$ and $R(H)_h$ the representation ring of H and its localization along h, respectively.

For an $R(T_n)$-module M, let $\Psi(M) \subset T_n$ denote the set of $R(T_n)$-weights of M. Moreover, we define $M[s] := R(T_n)_s \otimes_{R(T_n)} M$ and

$$\mathsf{ch} M := \sum_{s \in T_n} \dim M[s] \langle s \rangle \in \mathbb{Z} \langle T_n \rangle ,$$

where $\mathbb{Z} \langle T_n \rangle$ is a formal linear combination of elements of T_n.

We set

$$T_n(\vec{q}) := \{ s \in T_n(\mathbb{R}) \mid \epsilon_i(s) \in q_1 q^{\mathbb{Z}} \text{ for each } i = 1, \ldots, n \}.$$

For every $s \in T_n(\vec{q})$, we define $v_s \in \mathfrak{S}_n$ as the minimal length element such that

$$v_s s := \mathrm{Ad}(\dot{v_s}) s \in T_n(\vec{q}) \text{ satisfies } \epsilon_1(v_s s) \geq \epsilon_2(v_s s) \geq \cdots \geq \epsilon_n(v_s s).$$

A marked partition $\tau =(\mathbf{J}, \delta)$ of n is a pair consisting of a collection $\mathbf{J} = \{J_1, J_2, \ldots\}$ and a function $\delta : \{1, \ldots, n\} \to \{0, 1\}$ which satisfies

$$\bigsqcup_{j \geq 1} J_j = \{1, \ldots, n\}, \text{ and } \delta(i) = 1 \text{ for at most one } i \in J \text{ for each } J \in \mathbf{J}.$$

For simplicity, we may denote $J_j \in \tau$ instead of $J_j \in \mathbf{J}$. For a marked partition $\tau = (\mathbf{J}, \delta)$, we define $\mathbf{v}_\tau = \mathbf{v}^1_\tau \oplus \mathbf{v}^2_\tau$ with

$$\mathbf{v}^1_\tau = \sum_{i \in \{1, \ldots, n\}} \delta(i) \mathbf{v}_i \text{ and } \mathbf{v}^2_\tau = \sum_{J \in \mathbf{J}} \sum_{i, j \in J} \delta_1(\# \{ k \in J \mid i \leq k < j \}) \mathbf{v}_{i,j},$$

where $\delta_1(i) = 1$ $(i = 1)$ or 0 $(i \neq 1)$, and $\mathbf{v}_i \in V^{(1)}, \mathbf{v}_{i,j} \in V^{(2)}$ are T-eigenvectors of weights $\epsilon_i, \epsilon_i - \epsilon_j$, respectively. We put $\mathcal{O}_\tau := G \mathbf{v}_\tau \subset \mathbb{V}$.

We set $G(\chi) = G(s) := Z_G(s)$. A marked partition τ is adapted to $a = (s, \vec{q})$ or s if we have $s \mathbf{v}_\tau = q_1 \mathbf{v}^1_\tau \oplus q \mathbf{v}^2_\tau$. We set $\mathsf{P}_n(\vec{q})$ as the set of pairs $\chi = (s, \tau)$ consisting of $s \in T(\vec{q})$ and a marked partition τ adapted to s. For $J \in \tau$, we put $\underline{J} := \{ \epsilon_j(s) \mid j \in J \}$, which we regard as a ($q$-)segment. We write $I \in \chi$ if $I = \underline{J}$ for some $J \in \tau$. We set $\mathcal{O}_\chi := \dot{v_s} G(s) \mathbf{v}_\tau$. Two marked partitions τ, τ' adapted to s are called equivalent (and we denote this by $\tau \sim \tau'$) if

$$\mathcal{O}_{(s,\tau)} = \mathcal{O}_{(s,\tau')}. \tag{2.3}$$

This notion of equivalence can be translated in combinatorial terms on marked partitions; details are found in [7] §1.4.

Two parameters χ, χ' are called nested to each other if $I \Subset I'$ or $I' \Subset I$ holds for each $(I, I') \in \chi \times \chi'$.

For $\chi \in \mathsf{P}_n(\vec{q})$, let us denote by $\mathcal{E}_\chi$ the projection of $\mu_n^{-1}(\mathbf{v}_\tau)^a$ to G_n/B_n. Then, $M_\chi := H_\bullet(\mathcal{E}_\chi)$ admits a structure of a module over the specialized algebra $\mathbb{H}_a = \mathbb{H}_s := \mathbb{H}_n \otimes_{Z(\mathbb{H}_n)} \mathbb{C}_s$. We call M_χ the standard module attached to χ (cf. [12]). We denote the irreducible $\mathbb{H}_a$-module corresponding to χ by L_χ, which is a quotient of M_χ. We have a disjoint decomposition

$$\mathcal{E}_\chi = \bigsqcup_{s' \in W s \subset T_n} \mathcal{E}_\chi[s'], \text{ which induces } M_\chi = \bigoplus_{s' \in W s \subset T_n} M_\chi[s'] = \bigoplus_{s' \in W s \subset T_n} H_\bullet(\mathcal{E}_\chi[s']).$$

Let $\mathfrak{M}^n_{\vec{q}} \subset \mathfrak{Mod}^n_{\vec{q}}$ denote the fullsubcategory generated by simple modules corresponding to $\mathsf{P}_n(\vec{q})$. This is the category of $\mathbb{H}_{n,m}$-modules with central characters in $T(\vec{q})$ (cf. [12]). We denote by $K(\mathfrak{M}^n_{\vec{q}})$ its Grothendieck group.

We put $\mathsf{P}(\vec{q}) := \cup_{n\geq 1}\mathsf{P}_n(\vec{q})$. We have a natural map $\mathsf{P}(\vec{q}) \mapsto \mathsf{Q}(q)$ sending a pair (s,τ) with $\tau = (\mathbf{J},\delta)$ to $\{\underline{J} \mid J \in \mathbf{J}\}$. We sometimes identify $\mathbf{I} \in \mathsf{Q}(q)$ with its preimage in $\mathsf{P}(\vec{q})$ with trivial markings. We denote the set of such preimages by $\mathsf{P}^0(\vec{q})$. We denote the size of a marked partition by $|\tau|$ (or $|\chi|$).

Similarly, for $\mathbf{I} \in \mathsf{Q}(q)$, we denote the corresponding standard and irreducible $\mathbb{H}_n^{\mathsf{A}}$-modules by $M_{\mathbf{I}}^{\mathsf{A}}$ and $L_{\mathbf{I}}^{\mathsf{A}}$, respectively. For a segment I, we define its transpose to be the segment ${}^{\mathtt{t}}I = \{b^{-1} \mid b \in I\}$. For a multisegment $\mathbf{I}$, we define its transpose ${}^{\mathtt{t}}\mathbf{I}$ to be the multisegment $\{{}^{\mathtt{t}}I \mid I \in \mathbf{I}\}$ (with multiplicity counted). We sometimes denote $M_{{}^{\mathtt{t}}\mathbf{I}}^{\mathsf{A}}$ or $L_{{}^{\mathtt{t}}\mathbf{I}}^{\mathsf{A}}$ by ${}^{\mathtt{t}}M_{\mathbf{I}}^{\mathsf{A}}$ and ${}^{\mathtt{t}}L_{\mathbf{I}}^{\mathsf{A}}$, respectively.

We define ${}^0\mathfrak{Mod}_{\vec{q}}^n$ to be the category of finite-dimensional $\mathbb{H}_n^{\mathsf{A}}$-modules with positive real central characters. For each $\nu \in \mathbb{R}$, let St_n^{ν} denote the (central) twists of Steinberg representation of $\mathbb{H}_n^{\mathsf{A}}$ so that the corresponding unique segment is $I_n^{\nu} := \{q^{\nu}, q^{\nu+1}, \ldots, q^{\nu+n-1}\}$. We denote the central character of St_n^{ν} by s_n^{ν} and the central character of ${}^{\mathtt{t}}\mathsf{St}_n^{\nu} = {}^{\mathtt{t}}(\mathsf{St}_n^{\nu})$ by $\bar{\mathsf{s}}_n^{\nu}$.

We have an exact functor

$$ {}^0\mathfrak{Mod}_{\vec{q}} \times \mathfrak{Mod}_{\vec{q}} \ni (M_1, M_2) \mapsto M_1 \circledast M_2 \in \mathfrak{Mod}_{\vec{q}} $$

given by the parabolic induction. By abuse of notation, we also denote the parabolic induction of type A affine Hecke algebras as

$$ {}^0\mathfrak{Mod}_{\vec{q}} \times {}^0\mathfrak{Mod}_{\vec{q}} \ni (M_1, M_2) \mapsto M_1 \circledast M_2 \in {}^0\mathfrak{Mod}_{\vec{q}}. $$

For $\chi_i = (s_i, \mathbf{J}_i, \delta_i) \in \mathsf{P}_{n_i}(\vec{q})$ $(i = 1, 2)$, we set $\chi_1 \oplus \chi_2 := (s_1 \times s_2, \mathbf{J}_1 \sqcup \mathbf{J}_2[n_1], \delta_{12})$, where $\mathbf{J}_2[n_1]$ is the collection of subsets of $\{n_1 + 1, \ldots, n_1 + n_2\}$ obtained from $\mathbf{J}_2$ by uniformly adding n_1, and δ_{12} is the marking such that $\delta_{12} \mid_{\mathbf{J}_1} = \delta_1$ and $\delta_{12}(k) \mid_{\mathbf{J}_2[n_1]} = \delta_2(k - n_1)$ for each k.

2.4 Quotients of parabolic induction

The goal of this subsection is Proposition 2.10, which gives necessary conditions for an irreducible $\mathbb{H}_{n,m}$-module to appear as the quotient of a parabolically induced module. Before we prove this result, we need to fix notation and recall known results about quiver representations of type A. Throughout this subsection, we assume that m is *generic*.

For $\chi = (s,\tau) \in \mathsf{P}(\vec{q})$ such that $\tau = (\mathbf{J},\delta)$, we associate $\chi^0 = (s,\tau^0) \in \mathsf{P}^0(\vec{q})$ with $\tau^0 = (\mathbf{J}, 0)$.

Let $W[\chi]$ be the set of elements w of $W_{|\chi|}$ such that $w^{-1}s^{-1} \in \Psi(L_\chi)$. For $\vec{n} = (n_1, n_2)$, $n = n_1 + n_2$ with $n_1, n_2 \geq 0$, we define $\mathfrak{S}^{\vec{n}}$ (resp. $W^{\vec{n}}$) as a set of minimal length representative of $\mathfrak{S}_n/(\mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2})$ (resp. $W_n/(\mathfrak{S}_{n_1} \times W_{n_2})$) inside $\mathfrak{S}_n$ (resp. W_n).

For $\chi' = (s',\tau') \in \mathsf{P}(\vec{q})$, we say $\chi \leq \chi'$ if and only if $v_s s = v_{s'} s'$ and $\mathcal{O}_\chi \subset \overline{\mathcal{O}_{\chi'}}$. We refer to this (partial) ordering as the closure ordering. We define

$$ W[\chi]^{\circ} := W[\chi] - \bigcup_{\chi' > \chi} W[\chi']. $$

For a pair $(\chi_1, \chi_2) \in \mathsf{Q}(q) \times \mathsf{P}(\vec{q})$, we define

$$ W[\chi_1, \chi_2] := \{(w_1 \times w_2) \in W_{|\chi_1|+|\chi_2|} \mid w_i \in W[\chi_i]^{\circ}\}. \tag{2.4} $$

Lemma 2.1. *For each $\chi \in \mathsf{P}(\vec{q})$, we have $W[\chi]^{\circ} \neq \emptyset$. Moreover, we have $\mathfrak{S}_n \cap W[\chi]^{\circ} \neq \emptyset$ if $\chi = {}^{\mathtt{t}}\mathsf{St}_n^{\nu+m}$ for some $\nu \in \mathbb{Z}$.*

Proof. For the first assertion, it is enough to choose $w \in W$ so that the conditions of [7] Proposition 4.9 are satisfied, and this is straight-forward. The second assertion is also straight-forward since χ corresponds to a regular nilpotent orbit in $\mathfrak{gl}_n$. $\square$

For $s \in T_n(\vec{q})$, let $\mathbf{E}^s(i)$ denote the s-eigenspace of $V_n^{(1)}$ with its eigenvalue $q_1 q^i$. We have a natural identification

$$\mathbb{V}_n^{(s,\vec{q})} \cong \mathbf{E}^s(0) \oplus \mathbf{Rep}^s, \text{ where } \mathbf{Rep}^s = \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}(\mathbf{E}^s(i), \mathbf{E}^s(i+1)),$$

compatible with the $G_n(s)$-action.

For each $w \in W_n$, we set ${}^w\mathbb{V}_n^+ = \dot{w}^{-1}\mathbb{V}_n^+$, and let us denote by $\mathbf{Rep}_w^s$ the image of $({}^w\mathbb{V}_n^+ \cap \mathbb{V}_n^{(v_s s,\vec{q})})$ in $\mathbf{Rep}^s$ under the projection map $\mathbb{V}^{(s,\vec{q})} \to \mathbf{Rep}^s$. By abuse of notation, in place of $\mathbf{E}^s, \mathbf{Rep}^s, \ldots$, we may write $\mathbf{E}^\chi, \mathbf{Rep}^\chi, \ldots$ when we have a parameter $\chi = (s,\tau)$.

For each $w \in W_n$ and $s \in T_n(\vec{q})$, we define τ_w^s to be a marked partition adapted to $v_s s$ so that

$$\mathcal{O}_{\tau_w^s} \cap {}^w\mathbb{V}_n^+ \cap \mathbb{V}_n^{(v_s s,\vec{q})} \subset {}^w\mathbb{V}_n^+ \cap \mathbb{V}_n^{(v_s s,\vec{q})} \subset \mathbf{E}^s(0) \oplus \mathbf{Rep}^s$$

is open dense. It is clear that τ_w^s is well-defined up to equivalence (since there are only finitely many $G(s)$-orbits in $\mathbb{V}^{(s,\vec{q})}$). We set $\chi_w := (v_s s, \tau_w^s)$ for $\chi = (s,\tau) \in \mathsf{P}(\vec{q})$. (Note that χ_w depends on s and w, but not on τ.)

For $\chi = (s,\tau) \in \mathsf{P}(\vec{q})$, we set

$$\rho_{ij}(\chi) := \#\{I \in \chi \mid q_1 q^i, q_1 q^j \in I\} \text{ for every } j > i.$$

Theorem 2.2 (Abeasis-Del Fra [2], Zelevinsky [31])**.** *For each $\chi = (s,\tau) \in \mathsf{P}(\vec{q})$, the collection $\{\rho_{ij}(\chi)\}_{i,j}$ determines τ^0 uniquely. Moreover, we have*

1. *$\mathcal{O}_{(\chi_w)^0} \subset \overline{\mathcal{O}_{\chi^0}}$ if and only if $\dim A^{j-i}(\mathbf{E}^s(i)) \le \rho_{ij}(\chi)$ for every $A \in \mathbf{Rep}_w^s$;*
2. *If 1) holds, then we have $\mathcal{O}_{\chi^0} = \mathcal{O}_{(\chi_w)^0}$ if and only if some $A \in \mathbf{Rep}_w^s$ attains all the equalities in the condition 1).*

Moreover, we have $\mathcal{O}_\chi \subset \overline{\mathcal{O}_{\chi'}}$ only if $|\chi| = |\chi'|$ and $\rho_{ij}(\chi) \le \rho_{ij}(\chi')$ for every i,j.

Definition 2.3 (Elementary modification)**.** Let $\tau = (\mathbf{J}, \delta)$ be a marked partition adapted to s. For $J_1, J_2 \in \mathbf{J}$, we define another marked partition $\varepsilon_{J_1,J_2}(\tau) := (\mathbf{J}', \delta')$ as the maximal marked partition (with respect to the closure ordering) adapted to s which satisfies:

$$\mathbf{J}^\circ := \mathbf{J} - \{J_1, J_2\} \subset \mathbf{J}', \delta(J_1 \cup J_2) = \delta'(\mathbf{J}' - \mathbf{J}^\circ), \text{ and } \mathcal{O}_{(s,\tau)} \subsetneq \overline{\mathcal{O}_{(s,\varepsilon_{J_1,J_2}(\tau))}}.$$

Since both τ and $\varepsilon_{J_1,J_2}(\tau)$ are adapted to s, we put $\varepsilon_{\underline{J_1},\underline{J_2}}(\chi) := (s, \varepsilon_{J_1,J_2}(\tau))$ if $\chi = (s,\tau)$.

Lemma 2.4. *Keep the setting of Definition 2.3. If $\{J_1', J_2'\} = \mathbf{J}' - \mathbf{J}^\circ$, then we have*

- *$\underline{J_1'} = \underline{J_1} \cup \underline{J_2}$ and $\underline{J_2'} = \underline{J_1} \cap \underline{J_2}$ by swapping J_1' and J_2' if necessary;*
- *J_i' $(i = 1,2)$ is marked if and only if $q_1 \in \underline{J_i'}$ and $\delta(J_1 \cup J_2) = \{0,1\}$.*

Proof. Straight-forward from Theorem 2.2 and Definition 2.3. $\square$

The following is a reformulation of results from [2, 3]:

Theorem 2.5 (Abeasis-Del Fra-Kraft)**.** *For each $\chi = (s,\tau) \in \mathsf{P}^0(\vec{q})$ and $I, I' \in \chi$, we set $\chi' := \varepsilon_{I,I'}(\chi)$ (Definition 2.3). Assume $\chi \neq \chi'$ and $\max I < \max I'$, and we set $I^\flat := I \cap I'$ (if $I \cap I' \neq \emptyset$) or $\{\max I, \min I'\}$ (if $q \max I = \min I'$).*

1. *We have* $\dim \mathcal{O}_{\chi'} = \dim \mathcal{O}_{\chi} + 1$ *if there exists no* $I'' \in \chi'$ *such that*

$$I^{\flat} \subsetneq I'' \subsetneq I \cup I', \text{ or } I'' = I, I', \{\max I, \min I'\}.$$

2. *If 1) holds, then* $\overline{\mathcal{O}_{\chi'}}$ *is regular along* $\mathcal{O}_{\chi}$ *and the defining equation is locally given as:*

$$f \in \mathrm{Hom}(\wedge^k \mathbf{E}^s(i), \wedge^k \mathbf{E}^s(j))^*, \tag{2.5}$$

where $q_1 q^i = \min I^{\flat}, q_1 q^j = q \max I^{\flat}$*, and* $k := \rho_{ij}(\chi')$*.*

We also need the following result.

Lemma 2.6 ([7] Corollary 4.10)**.** *The map*

$$K(\mathfrak{M}^n_{\vec{q}}) \ni M \mapsto \mathsf{ch} M \in \mathbb{Z}\langle T_n \rangle$$

is an injection. □

Proposition 2.7. *Let* $(\chi_1, \chi_2) \in \mathsf{P}^0(\vec{q}) \times \mathsf{P}(\vec{q})$ *with* $n = |\chi_1| + |\chi_2|$*. We have*

$$\mathsf{ch} M_{\chi_1 \oplus \chi_2} = \mathsf{ch}(M^{\mathsf{A}}_{\chi_1} \circledast M_{\chi_2}) = \mathsf{ch}({}^{\mathsf{t}}M^{\mathsf{A}}_{\chi_1} \circledast M_{\chi_2}).$$

Proof. Let us denote $n_i = |\chi_i|$ $(i = 1, 2)$. We set $\mathbf{T} := T_n$. Let $\mathbf{P} \supset B_n$ be the parabolic subgroup of G_n with its reductive part $\mathbf{L} = GL(n_1) \times Sp(2n_2)$. Define $W_{\mathbf{L}} := N_{\mathbf{T}}(\mathbf{L})/\mathbf{T} \subset W$. We write $\chi_i := (s_i, \tau_i) = (s_i, \mathbf{J}^i, \delta^i)$ $(i = 1, 2)$, where $\delta^1 \equiv \{0\}$. We set $\mathbf{v} := \mathbf{v}_{\tau_1 \oplus \tau_2}$ and $\mathbf{v}_i := \mathbf{v}_{\tau_i}$ for $i = 1, 2$. We have $\mathbf{v} = \mathbf{v}_1 \oplus \mathbf{v}_2$. Let $\mathbf{r} \in T_{n_1} \cong T_{n_1} \times \{1\} \subset T_n$ be the element such that $\epsilon_i(\mathbf{r}) = r > 1$ for every $1 \le i \le n_1$ (and $= 1$ otherwise). Then, we have $\mathbf{r}\mathbf{v} = \mathbf{v}$ and hence $\mathbf{r}$ acts on $\mathcal{E}_{\chi_1 \oplus \chi_2}$.

Here we have $\mathbf{v}_1 \in \mathfrak{gl}_{n_1} = \mathfrak{gl}_{n_1} \oplus \{0\} \subset \mathbb{V}^{\mathbf{r}}$. Let $\mathcal{B}_{\mathbf{v}_1}$ be the type $\mathsf{A}_{n_1 - 1}$ Springer fiber of $\mathbf{v}_1$. We have

$$(\mathcal{E}_{\chi_1 \oplus \chi_2})^{\mathbf{r}} = \bigsqcup_{w \in W^{\vec{n}}} ((\mathcal{E}_{\chi_1 \oplus \chi_2})^{\mathbf{r}} \cap P\dot{w}^{-1}B/B) \cong \bigsqcup_{w \in W^{\vec{n}}} \mathcal{B}^{s_1}_{\mathbf{v}_1} \times \mathcal{E}_{\chi_2}. \tag{2.6}$$

Thanks to [6] §8.2, it follows that each $H_{\bullet}((\mathcal{E}_{\chi_1 \oplus \chi_2})^{\mathbf{r}} \cap P\dot{w}^{-1}B/B)$ admits an $R(\mathbf{T})$-module structure with

$$\mathsf{ch} H_{\bullet}((\mathcal{E}_{\chi_1 \oplus \chi_2})^{\mathbf{r}} \cap P\dot{w}^{-1}B/B) = w \mathsf{ch} H_{\bullet}((\mathcal{E}_{\chi_1 \oplus \chi_2})^{\mathbf{r}} \cap P/B).$$

By [12] Theorem 6.2, we conclude that

$$\begin{aligned} \mathsf{ch} M_{\chi_1 \oplus \chi_2} &= \sum_{w \in W^{\vec{n}}} \mathsf{ch} H_{\bullet}((\mathcal{E}_{\chi_1 \oplus \chi_2})^{\mathbf{r}} \cap P\dot{w}^{-1}B/B) \\ &= \sum_{w \in W^{\vec{n}}} w \mathsf{ch} H_{\bullet}((\mathcal{E}_{\chi_1 \oplus \chi_2})^{\mathbf{r}} \cap P/B) = \mathsf{ch}(M^{\mathsf{A}}_{\chi_1} \circledast M_{\chi_2}). \end{aligned}$$

The case $M^{\mathsf{A}}_{\chi_1}$ replaced with ${}^{\mathsf{t}}M^{\mathsf{A}}_{\chi_1}$ is similar. □

Corollary 2.8. *Keep the setting of Proposition 2.7. Let* L *be an irreducible* $\mathbb{H}_n$*-module. Then, we have*

$$[M_{\chi_1 \oplus \chi_2} : L] = [M^{\mathsf{A}}_{\chi_1} \circledast M_{\chi_2} : L] = [{}^{\mathsf{t}}M^{\mathsf{A}}_{\chi_1} \circledast M_{\chi_2} : L].$$

Proof. Combine Proposition 2.7 and Lemma 2.6. □

Lemma 2.9. *Let $\chi = (s, \tau) \in \mathsf{P}_{n_2}(\vec{q})$. Let ν' be an integer and set $\nu := m + \nu'$. For every $w_1 \times w_2 \in W[\mathsf{St}^{\nu}_{n_1}, \chi] \cap (\mathfrak{S}_{n_1} \times W_{n_2})$, we put $w = v_{\mathsf{s}^{\nu}_{n_1} \times s}(w_1 \times w_2)$ and $\chi' := (\mathsf{St}^{\nu}_{n_1} \oplus \chi)_w$. Then, we have*

$$\rho_{ij}(\mathsf{St}^{\nu}_{n_1}) + \rho_{ij}(\chi) \le \rho_{ij}(\chi') \le \rho_{ij}(\chi) + 1 \tag{2.7}$$

and $\rho_{ij}(\chi') = \rho_{ij}(\chi) + 1$ only if $\nu' \le j < \nu' + n_1$. If we replace $\mathsf{St}^{\nu}_{n_1}$ by ${}^{\mathsf{t}}\mathsf{St}^{\nu}_{n_1}$, then the inequalities (2.7) remain the same, and $\rho_{ij}(\chi') = \rho_{ij}(\chi) + 1$ only if $\nu' \le i < \nu' + n_1$.

Proof. We drop the superscripts ν during this proof. By construction, the natural $G(\mathsf{s}_{n_1}) \times G(\chi)$-equivariant embedding

$$\mathbf{Rep}^{\mathsf{s}_{n_1}} \oplus \mathbf{Rep}^{\chi} \hookrightarrow \mathbf{Rep}^{\chi'}$$

induces an embedding (of linear spaces which preserves compositions)

$$\mathbf{Rep}^{\mathsf{s}_{n_1}}_{w_1} \oplus \mathbf{Rep}^{\chi}_{w_2} \hookrightarrow \mathbf{Rep}^{\chi'}_{w}.$$

It follows that $\rho_{ij}(\mathsf{St}_{n_1}) + \rho_{ij}(\chi) \le \rho_{ij}(\chi')$ for every i, j. Moreover, the condition on w asserts that

$$\mathrm{Hom}(\mathbf{E}^{\mathsf{s}_{n_1}}(i), \mathbf{E}^{\chi}(i+1)) \cap \mathbf{Rep}^{\chi'}_{w} = \{0\} \text{ for every } i \in \mathbb{Z}. \tag{2.8}$$

It follows that every $A \in \mathbf{Rep}^{\chi'}_{w}$ preserves $\bigoplus_{i \in \mathbb{Z}} \mathbf{E}^{\chi}(i) \subset \bigoplus_{i \in \mathbb{Z}} \mathbf{E}^{\chi'}(i)$. Moreover, the image of the induced map $\mathbf{Rep}^{\chi'}_{w} \longrightarrow \mathbf{Rep}^{\chi}$ is contained in $\mathbf{Rep}^{\chi}_{w_2}$. Since $\dim \mathbf{E}^{\mathsf{s}_{n_1}}(i) \le 1$ for every $i \in \mathbb{Z}$, we conclude $\rho_{ij}(\chi') \le \rho_{ij}(\chi) + 1$. This proves the first part of the assertion.

To prove the second assertion, it suffices to see $A^{j-i}(\mathbf{E}^{\mathsf{s}_{n_1}}(i)) = \{0\}$ when $j \ge \nu' + n_1$. This follows by (2.8). For the case St replaced by ${}^{\mathsf{t}}\mathsf{St}$, we apply the same argument except for

$$\mathrm{Hom}(\mathbf{E}^{\chi}(i), \mathbf{E}^{\bar{\mathsf{s}}_{n_1}}(i+1)) \cap \mathbf{Rep}^{\chi'}_{w} = \{0\} \text{ for every } i \in \mathbb{Z}.$$

instead of (2.8). □

In view of Lemma 2.9, let us define

$$\mathcal{S}^{+}_{\nu,n_1}(\chi)^{\sim} := \left\{ \chi' = (s', \tau') \mid \begin{array}{c} s' = v_{\mathsf{s}^{\nu}_{n_1} \times s}(\mathsf{s}^{\nu}_{n_1} \times s), \rho_{ij}(\chi') - \rho_{ij}(\chi) \le 1, \text{ and} \\ \rho_{ij}(\chi') = \rho_{ij}(\chi) + 1 \text{ only if } \nu \le j < \nu + n_1 \end{array} \right\}$$

$$\mathcal{S}^{-}_{\nu,n_1}(\chi)^{\sim} := \left\{ \chi' = (s', \tau') \mid \begin{array}{c} s' = v_{\mathsf{s}^{\nu}_{n_1} \times s}(\mathsf{s}^{\nu}_{n_1} \times s), \rho_{ij}(\chi') - \rho_{ij}(\chi) \le 1, \text{ and} \\ \rho_{ij}(\chi') = \rho_{ij}(\chi) + 1 \text{ only if } \nu \le i < \nu + n_1 \end{array} \right\}.$$

Moreover, we define $\mathcal{S}^{\pm}_{\nu,n_1}(\chi) := \{\chi' \in \mathcal{S}^{\pm}_{\nu,n_1}(\chi)^{\sim} \mid \mathcal{O}_{\mathsf{St}^{\nu}_{n_1} \oplus \chi} \subset \overline{\mathcal{O}_{\chi'}}\}$.

Proposition 2.10. *Let $n = n_1 + n_2$ and ν' be natural numbers, and set $\nu := m + \nu'$. Let $\chi' \in \mathsf{P}_n(\vec{q})$. If we have a non-trivial surjection*

$$\mathsf{St}^{\nu}_{n_1} \circledast L_{\chi} \longrightarrow\!\!\!\!\!\rightarrow L_{\chi'} \text{ for some } \chi \in \mathsf{P}_{n_2}(\vec{q}),$$

then we have $\chi' \in \mathcal{S}^{+}_{\nu,n_1}(\chi)$. If we replace $\mathsf{St}^{\nu}_{n_1}$ with ${}^{\mathsf{t}}\mathsf{St}^{\nu}_{n_1}$, then the same statement holds only if $\chi' \in \mathcal{S}^{-}_{\nu,n_1}(\chi)$.

Proof. By the Frobenius reciprocity, it suffices to assume

$$\mathsf{St}^{\nu}_{n_1} \boxtimes L_{\chi} \hookrightarrow L_{\chi'} \tag{2.9}$$

as $\mathbb{H}^{\mathsf{A}}_{n_1} \otimes \mathbb{H}_{n_2}$-modules to deduce $\chi' \in \mathcal{S}^{+}_{\nu,n_1}(\chi)$. The condition (2.9) implies

$$\Psi(\mathsf{St}^{\nu}_{n_1} \boxtimes L_{\chi}) \subset \Psi(L_{\chi'}). \tag{2.10}$$

For every $w_1 \times w_2 \in (\mathfrak{S}_{n_1} \times W_{n_2} \cap W[\mathsf{St}^{\nu}_{n_1}, \chi])$, we have $s' = (w_1^{-1}\mathsf{s}^{\nu}_{n_1} \times w_2^{-1}s)^{-1} \in \Psi(\mathsf{St}^{\nu}_{n_1} \boxtimes L_{\chi})$. We set $\widetilde{\chi} := (\mathsf{St}^{\nu}_{n_1} \oplus \chi)_{v_{s'}(w_1 \times w_2)}$. By construction (cf. [7] §2.1), we have

$$\xi \in \mathsf{P}_n(\vec{q}) \text{ satisfies } L_{\xi}[s'] \neq \{0\} \Rightarrow \mathcal{O}_{\xi} \subset \overline{\mathcal{O}_{\widetilde{\chi}}}.$$

Hence, we conclude $\mathcal{O}_{\chi'} \subset \overline{\mathcal{O}_{\widetilde{\chi}}}$. By Lemma 2.9, this happens only if $\chi' \in \mathcal{S}^{+}_{\nu,n_1}(\chi)^{\sim}$. By Corollary 2.8, we have necessarily $[M_{\mathsf{St}^{\nu}_{n_1} \oplus \chi} : L_{\chi'}] \neq 0$ provided that (2.9) holds. This implies $\mathcal{O}_{\mathsf{St}^{\nu}_{n_1} \oplus \chi} \subset \overline{\mathcal{O}_{\chi'}}$, hence we conclude $\chi' \in \mathcal{S}^{+}_{\nu,n_1}(\chi)$. The other case is completely analogous. □

3 Delimits of tempered modules

In this section, we fix $m_0 \in \frac{1}{2}\mathbb{Z}$ and assume $|m - m_0| < \frac{1}{2}$. Let σ be a partition of n. As mentioned in the introduction, for every such m there is a central character $\mathsf{c}^{\sigma}_m \in T_n(\vec{q})$ and a single discrete series with parameter $\mathsf{ds}_m(\sigma) = (\mathsf{c}^{\sigma}_m, ds_m(\sigma))$; here $ds_m(\sigma)$ is a marked partition adapted to c^{σ}_m which parameterizes this discrete series. It might be helpful to visualize $\mathsf{ds}_m(\sigma)$ combinatorially as a left justified decreasing Young diagram coming from the partition σ by labeling every box with its m-content, i.e., the number $m + c(i,j)$, where $c(i,j)$ is the content $i - j$ of the box in the (i,j) position.

We sometimes identify $\mathsf{ds}_m(\sigma)$ with $L_{\mathsf{ds}_m(\sigma)}$. Let $\mathsf{mp}_m(\sigma)$ denote the anti-spherical parameter with central character c^{σ}_m. The $G(\mathsf{c}^{\sigma}_m)$-orbit indexed by $\mathsf{mp}_m(\sigma)$ is open dense.

3.1 Tempered delimits at generic parameter

We call a parameter $\chi \in \mathsf{P}(\vec{q})$ positive (resp. negative) if we have $\mathtt{E}_m(I) > 1$ (resp. < 1) for every $I \in \chi$.

Theorem 3.1 ([7] §3.3 + [12] Theorem 7.4)**.** *The parameter* $\mathsf{ds}_m(\sigma)$ *admits a unique decomposition* $\mathsf{ds}_m(\sigma) = \mathsf{ds}^{+}_m(\sigma) \oplus \mathsf{ds}^{-}_m(\sigma)$ *such that* $\mathsf{ds}^{+}_m(\sigma)$ *is positive and* $\mathsf{ds}^{-}_m(\sigma)$ *is negative. Moreover,*

1. $\mathsf{ds}^{-}_m(\sigma)$ *is not marked, and hence we can regard* $\mathsf{ds}^{-}_m(\sigma) \in \mathsf{Q}(q)$*;*
2. *two parameters* $\mathsf{ds}^{+}_m(\sigma)$ *and* $\mathsf{ds}^{-}_m(\sigma)$ *are nested to each other;*
3. *we have a surjection* $({}^{\mathsf{t}}L^{\mathsf{A}}_{\mathsf{ds}^{-}_m(\sigma)} \circledast L_{\mathsf{ds}^{+}_m(\sigma)}) \longrightarrow\!\!\!\!\!\rightarrow L_{\mathsf{ds}_m(\sigma)}$.

We denote by $ds^{\pm}_m(\sigma)$ the marked partitions corresponding to $\mathsf{ds}^{\pm}_m(\sigma)$, respectively.

Definition 3.2 (Delimits of tempered modules)**.** An algebraic flat family of irreducible $\mathbb{H}_{n,m}$-modules L^m depending on m ($m_0 - \frac{1}{2} < m < m_0 + \frac{1}{2}$ but $m \neq m_0$) with central character c^{σ}_m is called a delimit of tempered module, or just a tempered delimit if the limit $\lim_{m \to m_0} L^m$ is a tempered module. Let $\mathcal{D}_{m_0}(\sigma)$ be the set of isomorphism classes of irreducible tempered delimits with central character c^{σ}_m.

A segment I is called balanced along m_0 if $e_{+}(I)e_{-}(I) = q^{2(m-m_0)}$. A multisegment $\mathbf{I}$ is called balanced along m_0 if each member is a balanced segment along m_0.

Below, for a marked partition $\tau \in \mathcal{D}_{m_0}(\sigma)$ and a marked partition τ' obtained from τ, we denote the corresponding parameters by using bold letters. (I.e. $\boldsymbol{\tau} = (\mathsf{c}^{\sigma}_m, \tau)$, $\boldsymbol{\tau}'$, etc...).

Lemma 3.3. *Let* τ_m *denote a marked partition adapted to every* c^{σ}_m *(with* $m_0 < m < m_0 + \frac{1}{2}$*). Then* $\boldsymbol{\tau}_m$ *is the parameter of a tempered delimit if and only if there exists a multisegment* $\boldsymbol{\tau}^{\mathsf{e}}_m$ *and a marked partition* τ^{s}_m *such that the following conditions hold:*

- *We have a surjective map* $({}^{\mathsf{t}}L^{\mathsf{A}}_{\boldsymbol{\tau}^{\mathsf{e}}_m} \circledast L_{\boldsymbol{\tau}^{\mathsf{s}}_m}) \twoheadrightarrow L_{\boldsymbol{\tau}_m}$*;*
- $\boldsymbol{\tau}^{\mathsf{s}}_m = \mathsf{ds}_m(\sigma')$ *for some partition* σ' *obtained by removing* $\#\boldsymbol{\tau}^{\mathsf{e}}_m$ *hooks from* σ *(as Young diagrams);*
- $\boldsymbol{\tau}^{\mathsf{e}}_m$ *is a balanced multisegment along* m_0*;*
- *We have* $e_+(I) \neq e_+(I')$ *and* $e_-(I) \neq e_-(I')$ *for every pair* I, I' *of segments in* $\boldsymbol{\tau}^{\mathsf{e}}_m$*.*

Entirely the same statement holds if we replace m *with* $m_0 - \frac{1}{2} < m' < m_0$ *and* ${}^{\mathsf{t}}L^{\mathsf{A}}_{\boldsymbol{\tau}^{\mathsf{e}}_m}$ *with* $L^{\mathsf{A}}_{\boldsymbol{\tau}^{\mathsf{e}}_{m'}}$*.*

Proof. We prove only the case of ${}^{\mathsf{t}}L^{\mathsf{A}}_{\boldsymbol{\tau}^{\mathsf{e}}_m}$ since the case of $L^{\mathsf{A}}_{\boldsymbol{\tau}^{\mathsf{e}}_{m'}}$ is completely analogous. Let τ be a marked partition corresponding to a tempered delimit. By the Evens-Langlands classification [10], we have a unique quotient map

$$L^{\mathsf{A}} \circledast L_{\boldsymbol{\tau}^{\mathsf{s}}_m} \twoheadrightarrow L_{\boldsymbol{\tau}_m}, \tag{3.1}$$

where $L_{\boldsymbol{\tau}^{\mathsf{s}}_m}$ is a discrete series at $m_0 < m < m_0 + \frac{1}{2}$ for some smaller affine Hecke algebra of type C. Here (3.1) is a priori surjection for a specific value of m. Taking account into the fact that $\mathsf{P}(\vec{q})$ is constant for all the generic value of m and the Morita equivalences from [12] §9, we deduce that each element of $\mathsf{P}(\vec{q})$ defines an algebraic flat family of representations depending on m such that the function ch is continuous on m. Thanks to the uniqueness of quotients of Evens-Langlands induced modules (for each indivisual value), we deduce that (3.1) prolongs to an algebraic family depending on m and its quotient is irreducible for all values of $m_0 < m < m_0 + \frac{1}{2}$).

Assume $\tau^{\mathsf{s}}_m = (\mathbf{J}, \delta) = ds_m(\sigma')$, for some smaller partition σ'. The type A factor L^{A} is the unique quotient of the induction of

$$\mathsf{St}^{\nu'_1}_{k_1} \boxtimes \mathsf{St}^{\nu'_2}_{k_2} \boxtimes \cdots \boxtimes \mathsf{St}^{\nu'_p}_{k_p} \text{ with } \nu'_1 - \frac{k_1}{2} \le \nu'_2 - \frac{k_2}{2} \le \cdots \le \nu'_p - \frac{k_p}{2} \le \frac{-1}{2},$$

to $\mathbb{H}^{\mathsf{A}}_k$, where $k = \sum_{i=1}^p k_i$. By the Frobenius reciprocity, we have

$$\mathsf{St}^{\nu'_1}_{k_1} \boxtimes \mathsf{St}^{\nu'_2}_{k_2} \boxtimes \cdots \boxtimes \mathsf{St}^{\nu'_p}_{k_p} \boxtimes L_{\boldsymbol{\tau}^{\mathsf{s}}_m} \subset L_{\boldsymbol{\tau}_m} \tag{3.2}$$

as $R(T_n)$-modules. (Here $\nu'_1, \nu'_2, \ldots,$ are a priori real numbers.) In particular, the inclusion gives a nonzero map at the limit $m \to m_0$ (c.f. [7] §2.4). Therefore, we need $\lim_{m \to m_0} \nu'_i = (1 - k_i)/2$ for $i = 1, \ldots, p$ by the temperedness of the weights coming from the left hand side of (3.2). In particular, either $\mathsf{St}^{\nu'_i}_{k_i}$ or ${}^{\mathsf{t}}\mathsf{St}^{\nu'_i}_{k_i}$ corresponds to a m_0-balanced segment for each $1 \le i \le p$.

Since we fixed the central character to be c^{σ}_m, we conclude that $\nu'_i = (m_0 - m) + \frac{1-k_i}{2}$ for every $i = 1, 2, \ldots, p$. By induction-by-stages, we conclude

$${}^{\mathsf{t}}\mathsf{St}^{\nu_1}_{k_1} \circledast ({}^{\mathsf{t}}\mathsf{St}^{\nu_2}_{k_2} \circledast \cdots ({}^{\mathsf{t}}\mathsf{St}^{\nu_p}_{k_p} \circledast L_{\boldsymbol{\tau}^{\mathsf{s}}_m})) \twoheadrightarrow L_{\boldsymbol{\tau}_m}$$

with $\nu_i = (m - m_0) + \frac{1-k_i}{2}$ for $i = 1, \ldots, p$. In particular, the multisegment $\boldsymbol{\tau}^{\mathsf{e}}_m := \{\mathsf{St}^{\nu_1}_{k_1}, \mathsf{St}^{\nu_2}_{k_2}, \ldots, \mathsf{St}^{\nu_p}_{k_p}\}$ satisfies the first and the third conditions.

Claim A. *The partition* σ' *is obtained by removing a certain number of hooks from* σ *of length* $k_1, k_2, \ldots$*. In particular,* $k_1, k_2, \ldots, k_p$ *are distinct.*

Proof. Let $\widetilde{\sigma} = \{I_1, I_2, \ldots, I_\ell\}$ be a multisegment viewed as σ "unbent" along the diagonal. Notice that $q_1 \in I_1 \Subset I_2 \Subset \cdots \Subset I_\ell$. A hook in the original partition σ then becomes a path consisting of the union of a segment of the form $\{q_1, q_1 q, \ldots, e_+(I_i)\}$ with a segment of the form $\{e_-(I_j), qe_-(I_j), \ldots, q^{-1}q_1\}$ for some I_i, I_j. A subpartition $\widetilde{\sigma}' \subset \widetilde{\sigma}$ must satisfy $\widetilde{\sigma}' = \{I'_{i_1}, \ldots, I'_{i_p}\}$, $q_1 \in I_{i_1} \Subset \cdots \Subset I_{i_p}$, and $I'_{i_t} \subset I_{i_t}$. It is sufficient (by induction) to show that there exists a partition $\widetilde{\sigma}''$ such that $\widetilde{\sigma}' \subset \widetilde{\sigma}'' \subset \widetilde{\sigma}$, and σ'' is obtained from σ by removing one (balanced) hook. Set $\mathbf{S} = \widetilde{\sigma} \setminus \widetilde{\sigma}'$, and regard it as a balanced multisegment. We denote $S = \{b \in I \mid I \in \mathbf{S}\}$ and $S^{-1} = \{b^{-1} : b \in S\}$. Find the largest value $b_{\max}$ in $S \cup S^{-1}$. Assume this is in S (the other case is completely analogous). Then $b_{\max} = e_+(I_i)$ for some $I_i \in \widetilde{\sigma}$. Next find the smallest value in S, denote it $b_{\min}$. (Notice that $b_{\min}^{-1}$ is the largest values in S^{-1}.) Similarly, we must have $b_{\min} = e_-(I_j)$ for some $I_j \in \widetilde{\sigma}$. We claim that the segment $\{b_{\min}, \ldots, b_{\max}\}$, which is a hook, belongs to $\mathbf{S}$, or else there exist $b_-, b_+ \in \{b_{\min}, \ldots, b_{\max}\} \setminus S$ such that $\{b_{\min}, \ldots, q^{-1}b_-\} \in \mathbf{S}$ and $\{b_+ q, \ldots, b_{\max}\} \in \mathbf{S}$. But then it is clear that the segment $\{b_+ q, \ldots, b_{\max}\}$ cannot be balanced. □

We return to the proof of Lemma 3.3.

Thanks to Claim A, we deduce the second and the fourth conditions. Therefore, we have proved the "only if" part of the assertion.

We prove the "if" part of the assertion. We recall that $L^{\mathsf{A}}_{\boldsymbol{\tau}^{\mathsf{e}}_m}$, $L_{\boldsymbol{\tau}^{\mathsf{s}}_m}$, (and hence $L^{\mathsf{A}}_{\boldsymbol{\tau}^{\mathsf{e}}_m} \circledast L_{\boldsymbol{\tau}^{\mathsf{s}}_m}$) are algebraic families depending on m. Moreover, both of $L_{\boldsymbol{\tau}^{\mathsf{s}}_m}$ and $\lim_{m \to m_0} L^{\mathsf{A}}_{\boldsymbol{\tau}^{\mathsf{e}}_m}$ are tempered modules by assumptions 1) and 2), respectively. In particular, $\lim_{m \to m_0}(L^{\mathsf{A}}_{\boldsymbol{\tau}^{\mathsf{e}}_m} \circledast L_{\boldsymbol{\tau}^{\mathsf{s}}_m})$ is tempered, and hence all of its irreducible constituents are tempered. □

Corollary 3.4 (of the proof of Lemma 3.3)**.** *Keep the setting of Lemma 3.3. Let h be the number of hooks in σ which give balanced segments along m_0. Then, we have $\#\mathcal{D}_{m_0}(\sigma) = 2^h$.* □

3.2 The classification of tempered delimits

We define two subsets of $\mathcal{D}_{m_0}(\sigma)$ as:

$$\mathcal{D}^{\pm}_{m_0}(\sigma) := \{\tau \in \mathcal{D}_{m_0}(\sigma) \mid \boldsymbol{\tau}^{\mathsf{s}}_m = \mathsf{ds}^{\pm}_m(\sigma') \text{ for } \pm m_0 - \frac{1}{2} < \pm m < \pm m_0\},$$

where the $\pm$ denote a uniform choice of $+$ or $-$, and σ' is borrowed from Lemma 3.3.

Proposition 3.5. *Let $\tau^{\mathsf{e}}_m = (\mathbf{J}^{\mathsf{e}}, 0)$ and τ^{s}_m be the marked partitions obtained from $\tau_m \in \mathcal{D}_{m_0}(\sigma)$ by Lemma 3.3. Then there exists a marked partition $\tau^{\mathsf{f}}_m = (\mathbf{J}^{\mathsf{e}}, \delta^{\mathsf{f}})$ (obtained from τ^{e}_m by possibly changing the marking) and a decomposition $\tau_m = \tau^{\mathsf{f}}_m \oplus \tau^{\mathsf{s}}_m$ if one of the following conditions hold:*

- *$\tau \in \mathcal{D}^{+}_{m_0}(\sigma)$ and $m_0 - \frac{1}{2} < m < m_0$;*
- *$\tau \in \mathcal{D}^{-}_{m_0}(\sigma)$ and $m_0 < m < m_0 + \frac{1}{2}$.*

Proof. Since the two cases are completely analogous, we prove only the first case. We use the intermediate step of the ds_m-algorithm (in the sense of [7] Algorithm 3.3 step 2) to σ. It yields a sequence of segments

$$I_1, I_2, I_3, \ldots \text{ with } \max\{e_+(I_1), e_-(I_1)^{-1}\} > \max\{e_+(I_2), e_-(I_2)^{-1}\} > \cdots \tag{3.3}$$

so that each segment of $\mathsf{ds}_m(\sigma)$ is a union of at most two of them.

Let $\mathbf{I}^\circ$ be the set of all segments I such that I is obtained by gluing I^+ and I^- in (3.3) with the property that $e_+(I^+) = q^{2(m-m_0)} e_-(I^-)^{-1}$. Let τ_m^{e} and τ_m^{s} be the marked partitions from Lemma 3.3. We have $\boldsymbol{\tau}_m^{\mathsf{e}} \subset \mathbf{I}^\circ$. For each $I \in \mathbf{I}^\circ$, we have some $j(I)$ so that $I = I_{j(I)} \cup I_{j(I)+1}$. Notice that $e_+(I_{j(I)}) < e_+(I_{j(I)+1})$ by $m < m_0$. The assumption $\mathsf{ds}_m(\sigma') = \mathsf{ds}_m^+(\sigma')$ implies that we have $\mathtt{E}_m(I') > 1$ for every $I' \in \mathsf{ds}_m(\sigma')$. In particular, there exists no $I' \in \mathsf{ds}_m(\sigma')$ such that $I' \lhd I$.

Moreover, we have $I \Subset I''$ or $I'' \Subset I$ for every distinct $I, I'' \in \mathbf{J}^{\mathsf{e}}$. We set $\boldsymbol{\tau}_m^{\mathsf{e}} = \{I_1^{\mathsf{e}}, \ldots, I_N^{\mathsf{e}}\}$ and $\boldsymbol{\tau}_m^{\mathsf{s},k} := \{I_1^{\mathsf{e}}\} \oplus \{I_2^{\mathsf{e}}\} \oplus \cdots \oplus \{I_k^{\mathsf{e}}\} \oplus \boldsymbol{\tau}_m^{\mathsf{s}}$. Applying Proposition 2.10 to $\{I_{k-1}^{\mathsf{e}}\}$ and $\boldsymbol{\tau}_m^{\mathsf{s},k}$ for each $k = 1, \ldots, N$, we conclude that

$$L^{\mathsf{A}}_{I_1^{\mathsf{e}}} \circledast (\cdots (L^{\mathsf{A}}_{I_N^{\mathsf{e}}} \circledast L_{\boldsymbol{\tau}_m^{\mathsf{s}}})) \twoheadrightarrow L^{\mathsf{A}}_{\boldsymbol{\tau}_m^{\mathsf{e}}} \circledast L_{\boldsymbol{\tau}_m^{\mathsf{s}}} \twoheadrightarrow L_{\boldsymbol{\tau}_m^{\mathsf{f}} \oplus \boldsymbol{\tau}_m^{\mathsf{s}}}$$

for some $\boldsymbol{\tau}_m^{\mathsf{f}}$ obtained from $\boldsymbol{\tau}_m^{\mathsf{e}}$ by changing the markings if necessary. By the uniqueness of the quotient of the Evens-Langlands induction, we conclude that $\boldsymbol{\tau}_m = \boldsymbol{\tau}_m^{\mathsf{f}} \oplus \boldsymbol{\tau}_m^{\mathsf{s}}$ as required. □

Lemma 3.6. *There exists a unique decomposition $n = n_1 + n_2 + \cdots + n_p$, and a unique sequence $\sigma^1, \sigma^2, \cdots, \sigma^p$ of partitions of $n_1, n_2, \ldots, n_p$, with the following properties:*

1. *$I \Subset I'$ if $I \in \mathsf{mp}_m(\sigma^i)$ and $I' \in \mathsf{mp}_m(\sigma^j)$ for $i < j$;*
2. *$\mathsf{mp}_m(\sigma^i)$ is positive if and only if $\mathsf{mp}_m(\sigma^{i+1})$ is negative for each i;*
3. *$\mathsf{mp}_m(\sigma^i)$ is negative if and only if $\mathsf{mp}_m(\sigma^{i+1})$ is positive for each i.*

Proof. Let $\sigma_0^1, \sigma_0^2, \cdots, \sigma_0^{p'}$ be the sequence of partitions which gives the finest nested component decomposition

$$\mathsf{mp}_m(\sigma) = \mathsf{mp}_m(\sigma_0^1) \oplus \mathsf{mp}_m(\sigma_0^2) \oplus \cdots \oplus \mathsf{mp}_m(\sigma_0^{p'}).$$

By rearranging the order of the sequence if necessary, we can assume $I \Subset I'$ if $I \in \mathsf{mp}_m(\sigma_0^i)$ and $I' \in \mathsf{mp}_m(\sigma_0^j)$ for $i < j$. From [7], §3.5, we know that every nested component $\mathsf{mp}_m(\sigma) = \mathsf{mp}_m(\sigma^i)$ is either positive or negative. Hence, by joining σ_0^i with σ_0^{i+1} if both of $\mathsf{mp}_m(\sigma_0^i)$ and $\mathsf{mp}_m(\sigma_0^{i+1})$ are simultaneously positive or negative, we obtain the desired sequence of partitions. □

We refer the decomposition of σ into $\sigma^1, \sigma^2, \ldots, \sigma^p$ in Lemma 3.6 as the canonical decomposition of σ (with respect to the parameter m).

Lemma 3.7. *Fix the canonical decomposition $\sigma^1, \sigma^2, \cdots, \sigma^p$ of σ. Let h_i be the number of balanced hooks of σ^i and let h be the number of balanced hooks of σ (along m_0). Then, we have $h = \sum_i h_i$.*

Proof. Let I be a balanced segment obtained from a hook of σ. Then, there exists $I^+, I^- \in \mathsf{mp}_m(\sigma)$ such that $e_-(I) = e_-(I^-)$ and $e_+(I) = e_+(I^+)$. If $I^+ = I^-$, then we have $I^+ = I^- = I$. Hence, we have $I \in \mathsf{mp}_m(\sigma^i)$ for some i. Otherwise, we have $\mathtt{E}_{m_0}(I^+) \neq 1 \neq \mathtt{E}_{m_0}(I^-)$. To prove the assertion, it suffices to verify $I^+, I^- \in \mathsf{mp}_m(\sigma^i)$ for some i. We assume to the contrary to deduce contradiction. Then, we have either $I^+ \Subset I^-$ or $I^- \Subset I^+$. By inspection, we see that if $I^+ \Subset I^-$, then both I^+, I^- are positive. Similarly, if $I^- \Subset I^+$, then both are negative. Assume that $I^+ \Subset I^-$ (and hence they are positive). If there exists $I' \in \mathsf{mp}_m(\sigma)$ such that $I^+ \Subset I' \Subset I^-$, then we have $e_+(I') > e_+(I)$ and $e_-(I') > e_-(I)$. It follows that I' is automatically positive. Therefore, we conclude that $I^+, I^- \in \mathsf{mp}_m(\sigma^i)$ for some i by construction of σ^i. We have $I^+, I^- \in \mathsf{mp}_m(\sigma^i)$ for some i in the case $I^- \Subset I^+$ by a similar argument, which completes the proof. □

Definition 3.8. Let $\sigma^1, \sigma^2, \ldots, \sigma^p$ be the canonical decomposition of σ with respect to the parameter $m_0 < m < m_0 + \frac{1}{2}$. Let $\mathcal{C}_{m_0}(\sigma^k)$ $(1 \le k \le p)$ be the set of (equivalence class of) marked partitions τ_k adapted to $\mathsf{c}^m_{\sigma^k}$ which admit a decomposition

$$\tau_k = \tau_k^\sharp \oplus \tau_k^\flat \oplus \tau_k^+ \oplus \tau_k^- = \tau_k^\sharp \oplus \tau_k^\perp \tag{3.4}$$

$(\tau_k^\perp = \tau_k^\flat \oplus \tau_k^+ \oplus \tau_k^-)$ with the following properties:

1. $\tau_k^\sharp$ is the set of all unmarked $J \in \tau_k$ such that $\underline{J}$ is a balanced segment along m_0 obtained from a hook of σ^k;
2. $\tau_k^\flat$ is the set of all marked $J \in \tau_k$ such that $\underline{J}$ is a balanced segment along m_0 obtained from a hook of σ^k. We have $\tau_k^\flat = \emptyset$ if $\mathsf{mp}(\sigma^k)$ is negative;
3. $\tau_k^+ = ds_{m'}(\sigma^+) = ds^-_{m'}(\sigma^+)$ for some σ^+ and $m_0 < m' < m_0 + \frac{1}{2}$;
4. $\tau_k^- = ds_{m'}(\sigma^-) = ds^+_{m'}(\sigma^-)$ for some σ^- and $m_0 - \frac{1}{2} < m' < m_0$.

Then, we define

$$\mathcal{C}_{m_0}(\sigma) := \{\bigoplus_{k=1}^p \tau_k \mid \tau_k \in \mathcal{C}_{m_0}(\sigma^k) \text{ for } k = 1, \ldots, p\}. \tag{3.5}$$

We may sometimes identify $\tau \in \mathcal{C}_{m_0}(\sigma)$ with the corresponding parameter, which we denote by $\boldsymbol{\tau} := (\mathsf{c}^m_\sigma, \tau)$.

Lemma 3.9. *Keep the setting of Definition 3.8. For each $\tau_j \in \mathcal{C}_{m_0}(\sigma^j)$ and $\tau_l \in \mathcal{C}_{m_0}(\sigma^l)$ with $j \neq l$, the two parameters $\boldsymbol{\tau}_j$ and $\boldsymbol{\tau}_l$ are nested to each other. In addition, each $\tau \in \mathcal{C}_{m_0}(\sigma)$ admits a decomposition*

$$\tau = \tau^\sharp \oplus \tau^\flat \oplus \tau^+ \oplus \tau^- = \tau^\sharp \oplus \tau^\perp \tag{3.6}$$

with the properties 1)–4) (τ_k replaced with τ).

Proof. Since all balanced segments are obtained by gluing the intermediate output of the ds_m-algorithm as in the proof of Proposition 3.5, we deduce

$$\max\{e_+(I) \mid I \in \mathsf{mp}(\sigma^j)\} = \max\{e_+(I) \mid I \in \boldsymbol{\tau}_j\} \text{ and}$$
$$\min\{e_-(I) \mid I \in \mathsf{mp}(\sigma^j)\} = \min\{e_-(I) \mid I \in \boldsymbol{\tau}_j\}.$$

If $j > 1$, then we further have

$$\{e_+(I) \mid I \in \mathsf{mp}(\sigma^j)\} = \{e_+(I) \mid I \in \boldsymbol{\tau}_j\} \text{ and}$$
$$\{e_-(I) \mid I \in \mathsf{mp}(\sigma^j)\} = \{e_-(I) \mid I \in \boldsymbol{\tau}_j\}.$$

Therefore, we deduce

$$\{e_+(I) \mid I \in \boldsymbol{\tau}_j\} > \{e_+(I) \mid I \in \boldsymbol{\tau}_l\} \text{ and } \{e_-(I) \mid I \in \boldsymbol{\tau}_j\} < \{e_-(I) \mid I \in \boldsymbol{\tau}_l\}$$

whenever $j > l$. This implies the first assertion. Thanks to Lemma 3.6 and [7] Corollary 3.17, the second assertion follows from the first assertion. □

Corollary 3.10. *We have $\#\mathcal{D}_{m_0}(\sigma) \le \#\mathcal{C}_{m_0}(\sigma)$.*

Proof. Thanks to Lemma 3.7, Definition 3.8 and Lemma 3.9, it suffices to prove the assertion only when $\sigma = \sigma^1$ gives the canonical decomposition of σ with respect to $m_0 < m < m_0 + \frac{1}{2}$. Thus, we can assume that $\mathsf{mp}(\sigma)$ is either positive or negative. If $\mathsf{mp}(\sigma)$ is positive, then we deduce $\mathcal{D}_{m_0}(\sigma) \subset \mathcal{C}_{m_0}(\sigma)$ by Proposition 3.5.

If $\mathsf{mp}(\sigma)$ is negative, then we have $e_-(I)^{-1} \geq q^{1-\epsilon} e_+(I)$ for $\epsilon = 2(m - m_0) > 0$. We borrow notation $I_1, I_2, \ldots$ and $\mathbf{I}^\circ$ from the proof of Proposition 3.5. If $I \in \mathbf{I}^\circ$ is obtained as a union of $I_{j(I)}$ and $I_{j(I)+1}$, then $I_{j(I)}$ is glued with some I_k with $k < j(I)$. Therefore, we deduce that $\mathsf{ds}_m(\sigma') = \mathsf{ds}_m^-(\sigma')$ implies that $\mathsf{ds}_m(\sigma')$ does not contain a balanced segment along m_0. It follows that $\#\mathcal{C}_{m_0}(\sigma)$ is at least the cardinality of the power set of $\mathbf{I}^\circ$. Hence, we conclude the result by Corollary 3.4 in this case. $\square$

Definition 3.11. The decomposition (3.6) is unique if we rearrange the cardinality of $\tau^\sharp$ to be maximal in the equivalence class (in the sense of (2.3)). When this maximality condition is attained, we call (3.6) the standard decomposition (with respect to the parameter m_0).

Proposition 3.12. *Let $\tau, \tau' \in \mathcal{C}_{m_0}(\sigma)$. We have $\mathcal{O}_{\boldsymbol{\tau}'} \subset \overline{\mathcal{O}_{\boldsymbol{\tau}}}$ if and only if $\boldsymbol{\tau}^\sharp \subset (\boldsymbol{\tau}')^\sharp$ as multisegments (by using standard decomposition). Moreover, we have*

$$\dim \mathcal{O}_{\boldsymbol{\tau}} = \dim \mathcal{O}_{\boldsymbol{\tau}'} + \#(\boldsymbol{\tau}')^\sharp - \#\boldsymbol{\tau}^\sharp.$$

Proof. Thanks to Lemma 3.9 and [7] Corollary 2.10, it suffices to prove the assertion in the case $\mathsf{mp}_m(\sigma)$ consists of only positive (or only negative) nested components. By separating out nested components which satisfies $\mathsf{ds}_{m'}(\sigma') = \mathsf{ds}_m(\sigma')$ for $m_0 - \frac{1}{2} < m' < m_0 < m < m_0 + \frac{1}{2}$, there remain three cases to be considered: **0)** $\mathsf{mp}_m(\sigma)$ consists of a unique balanced segment, or

$$\begin{array}{lll} \mathbf{p}) & e_+(I) \geq q^{1+\epsilon} e_-(I)^{-1}, \text{ for every } I \in \mathsf{mp}_m(\sigma), \text{ or} & \\ \mathbf{n}) & e_+(I) \leq q^{-1+\epsilon} e_-(I)^{-1} \text{ for every } I \in \mathsf{mp}_m(\sigma), & (3.7) \end{array}$$

where $\epsilon = 2(m - m_0) > 0$.

Case **0)**: We have $\tau = (\{J\}, \delta)$ with a single J and different choice of δ from τ'. If $\delta \not\equiv 0$, then we have $\tau = \tau^\flat$ and if $\delta \equiv 0$, then we have $\tau = \tau^\sharp$. Hence, the assertion is straight-forward in this case.

Case **p)**: We assume that $\boldsymbol{\tau}^\sharp \cup \{I\} = (\boldsymbol{\tau}')^\sharp$ as multisegments. By the ds_m algorithm and condition **p)**, we deduce that there exists $I_* \in \boldsymbol{\tau}$ so that $I \lhd I_*$. By rearranging I_* if necessary, we can assume that $e_-(I_*) = \min\{e_-(I') \mid I' \in \boldsymbol{\tau}, I \lhd I'\}$. Notice that such I_* is unique since the minimal/maximal entries of segments of an output of the ds_m-algorithm are all distinct. We have $\boldsymbol{\tau} = \varepsilon_{I_*,I}(\boldsymbol{\tau}')$ by inspection. By the minimality assumption on I_*, there exists no segment $I' \in \boldsymbol{\tau} - \{I_*, I\}$ such that

$$I_* \cap I \subsetneq I' \subsetneq I_* \cup I.$$

By Theorem 2.5, we conclude that

$$\mathcal{O}_{\boldsymbol{\tau}'} \subset \overline{\mathcal{O}_{\boldsymbol{\tau}}} \text{ and } \dim \mathcal{O}_{\boldsymbol{\tau}} = \dim \mathcal{O}_{\boldsymbol{\tau}'} + 1.$$

We set $I = \{q_1 q^{m_-}, \cdots, q_1 q^{m_+}\}$. Then, we have

$$\begin{array}{ll} \rho_{m_-, m_+ + 1}(\boldsymbol{\tau}') = \rho_{m_-, m_+ + 1}(\boldsymbol{\tau}) - 1 \text{ and} & \\ \rho_{m_- - l, m_+ + 1}(\boldsymbol{\tau}') = \rho_{m_- - l, m_+ + 1}(\boldsymbol{\tau}) \text{ for every } l > 0. & (3.8) \end{array}$$

Let $\tau_0 \in \mathcal{C}_{m_0}(\sigma)$ be the marked partition obtained by setting $\tau_0^\sharp$ to be the collection of all hooks in σ which gives a balanced segment along m_0. We have $\mathcal{O}_{\boldsymbol{\tau}_0} \subset \overline{\mathcal{O}_{\boldsymbol{\tau}}} \cap \overline{\mathcal{O}_{\boldsymbol{\tau}'}}$. Notice that each pair of segments of $\boldsymbol{\tau}_0^\sharp$ are nested to each other. Therefore, a repeated use of (3.8) claims that

$$\rho_{m_-,m_++1}(\boldsymbol{\tau}'') = \begin{cases} \rho_{m_-,m_++1}(\boldsymbol{\tau}) - 1 & (I \in (\boldsymbol{\tau}'')^\sharp) \\ \rho_{m_-,m_++1}(\boldsymbol{\tau}) & (I \notin (\boldsymbol{\tau}'')^\sharp) \end{cases} \tag{3.9}$$

for every $\tau'' \in \mathcal{C}_{m_0}(\sigma)$. Therefore, we have $\mathcal{O}_{\tau'} \subset \overline{\mathcal{O}_\tau}$ only if $\boldsymbol{\tau}^\sharp \subset (\boldsymbol{\tau}')^\sharp$ by Theorem 2.2.

Case **n)**: The proof goes in a similar fashion if we replace "$I \lhd I_*$" by "$I_* \lhd I$", and min by max.

This case-by-case analysis implies the result as desired. □

Corollary 3.13. *Keep the setting of Proposition 3.12. Then $\overline{\mathcal{O}_{\boldsymbol{\tau}}}$ is smooth along $\mathcal{O}_{\boldsymbol{\tau}'}$.*

Proof. We assume the setting of the proof of Proposition 3.12 and denote $s := \mathsf{c}_m^\sigma$. The case **0)** is clear. We verify the assertion in the case **p)**. Then, every relevant $G(s)$-orbit is obtained as the pullback of an orbit of $\mathbf{Rep}^s$ to $(\mathbf{E}^s(0) \oplus \mathbf{Rep}^s)$. By Theorem 2.5 2), we have an algebraic function f_I on $(\mathbf{E}^s(0) \oplus \mathbf{Rep}^s)$ for each $I \in \boldsymbol{\tau}_0^\sharp$ such that

$$\mathcal{O}_{\boldsymbol{\tau}} \subset \{f_I = 0\} \text{ if } I \in \boldsymbol{\tau}^\sharp \text{ and } \overline{\mathcal{O}_{\boldsymbol{\tau}} \cap \{f_I \neq 0\}} = \overline{\mathcal{O}_{\boldsymbol{\tau}}} \text{ if } I \notin \boldsymbol{\tau}^\sharp.$$

Moreover, we have $df_I \not\equiv 0$ on $\mathcal{O}_{\boldsymbol{\tau}}$ by inspection. Therefore, $\{f_I; I \in (\boldsymbol{\tau}')^\sharp - \boldsymbol{\tau}^\sharp\}$ gives an algebraically independent system of equation of $\mathcal{O}_{\boldsymbol{\tau}'}$ along an open dense subset of $\mathcal{O}_{\boldsymbol{\tau}}$. In particular, $\mathcal{O}_{\boldsymbol{\tau}'}$ is locally a complete intersection inside $\overline{\mathcal{O}_{\boldsymbol{\tau}}}$. This is the very definition of smoothness. Hence we have verified the case **p)**. The case **n)** is similar. □

The proofs of the following three Theorems 3.14, 3.15, and 3.16 are simultaneously given in §3.3.

Theorem 3.14 (Classification of tempered delimits)**.** *We have an equality $\mathcal{C}_{m_0}(\sigma) = \mathcal{D}_{m_0}(\sigma)$, where $\mathcal{C}_{m_0}(\sigma)$ and $\mathcal{D}_{m_0}(\sigma)$ are as in (3.5) and Definition 3.2, respectively.*

Theorem 3.15. *Let $\tau \in \mathcal{D}_{m_0}(\sigma)$ and let $\tau = \tau^\sharp \oplus \tau^\perp$ be its standard decomposition as in Definition 3.11. Fix $\tau_\ominus \subset \tau^\sharp$ and consider an induced decomposition $\tau = \tau_\ominus \oplus \tau_\circledast$. We have*

$$[L^{\mathsf{A}}_{\boldsymbol{\tau}_\ominus} \boxast L_{\boldsymbol{\tau}_\circledast}] = \sum_{\tau' \in \mathcal{D}_{m_0}(\sigma); \tau_\circledast^\sharp \subset (\tau')^\sharp \subset \tau_\ominus \oplus \tau_\circledast^\sharp} [L_{\boldsymbol{\tau}'}] \in K(\mathfrak{M}_q^n), \tag{3.10}$$

where $\tau' = (\tau')^\sharp \oplus (\tau')^\perp$ is the standard decomposition of $\tau' \in \mathcal{D}_{m_0}(\sigma)$.

Theorem 3.16. *Let $\tau_0 \in \mathcal{D}_{m_0}(\sigma)$ be the minimal element with respect to the closure ordering (i.e. $\tau_0^\sharp$ is maximal). Then, $\lim_{m \to m_0} L_{\boldsymbol{\tau}_0^\perp}$ is an irreducible discrete series.*

Corollary 3.17 (of Theorem 3.15)**.** *Let $\tau \in \mathcal{D}_{m_0}(\sigma)$ and $m_0 < m < m_0 + \frac{1}{2}$. We have*

$$\tau_m^{\mathsf{e}} \oplus \tau_m^{\mathsf{s}} \in \mathcal{D}_{m_0}(\sigma).$$

In particular, (3.10) applied for $\tau_\ominus := \tau_m^{\mathsf{e}}$ and $\tau_\circledast := \tau_m^{\mathsf{s}}$ for each $\tau \in \mathcal{D}_{m_0}(\sigma)$ yields an overdetermined system of character equations.

In section 4, we need the following Corollary 3.19 of Theorem 3.15.

Definition 3.18. We say that τ, τ' in $\mathcal{D}_{m_0}(\sigma)$ are linked if there exist properly parabolically induced modules $V_1, \ldots, V_k$ such that in the Grothendieck group of $\mathbb{H}_{n,m}$ we have:

$$[L_{\boldsymbol{\tau}}] + [L_{\boldsymbol{\tau}'}] \text{ or } [L_{\boldsymbol{\tau}}] - [L_{\boldsymbol{\tau}'}] \in \mathrm{Span}_{\mathbb{Z}}([V_1], \ldots, [V_k]) \subset K(\mathfrak{M}^n_q).$$

Corollary 3.19. *Assume $m_0 - \frac{1}{2} < m < m_0 + \frac{1}{2}$ and $m \neq m_0$. Any two tempered delimits in $\mathcal{D}_{m_0}(\sigma)$ are linked (in the sense of Definition 3.18).*

Proof. We use induction on h, the number of balanced hooks at m, to show that there exists a system of 2^h distinct equations (in the Grothendieck group) of the form

$$[L_{\boldsymbol{\tau}_i}] + [L_{\boldsymbol{\tau}_j}] = [V_{ij}],$$

where V_{ij} is a properly parabolically induced modules, for $\tau_i, \tau_j \in \mathcal{D}_{m_0}(\sigma)$. Moreover, every $\tau \in \mathcal{D}_{m_0}(\sigma)$ appears exactly 2^{h-1} times in these equations. Since $2^{h-1} + 2^{h-1} = 2^h$, the claim follows. □

3.3 Proofs of Theorems 3.14, 3.15, and 3.16

We start with certain weaker versions of Theorems 3.16 and 3.15, which turn out to be sufficient in order to prove the full statements.

Lemma 3.20. *Let $\tau_0 \in \mathcal{C}_{m_0}(\sigma)$ be the minimal element with respect to the closure ordering (i.e. $\tau_0^\sharp$ is maximal). Then, $\lim_{m \to m_0} L_{\boldsymbol{\tau}_0^\perp}$ must be an irreducible discrete series.*

Proof. Let σ' be a partition of n' such that $\tau_0^\perp \in \mathcal{C}_{m_0}(\sigma')$. By the assumption $\tau_0^\sharp$ is maximal, we deduce that σ' does not contains a balanced hook along m_0. Hence, we have

$$\tau_0^\perp = ds_{m'}(\sigma') = ds_m(\sigma')$$

for $m_0 - \frac{1}{2} < m' < m_0 < m < m_0 + \frac{1}{2}$. In particular, $\lim_{m \to m_0} L_{\boldsymbol{\tau}_0^\perp}$ must be discrete series by Opdam-Solleveld [25]. □

Proposition 3.21. *Let $\tau \in \mathcal{C}_{m_0}(\sigma)$ and let $\tau = \tau^\sharp \oplus \tau^\perp$ be the standard decomposition. Fix $\tau_\ominus \subset \tau^\sharp$ and consider an induced decomposition $\tau = \tau_\ominus \oplus \tau_\circledast$. We have*

$$[L^{\mathsf{A}}_{\boldsymbol{\tau}_\ominus} \circledast L_{\boldsymbol{\tau}_\circledast}] = E + \sum_{\tau' \in \mathcal{D}_{m_0}(\sigma); \tau^\sharp_\circledast \subset (\tau')^\sharp \subset \tau_\ominus \oplus \tau^\sharp_\circledast} [L_{\boldsymbol{\tau}'}] \in K(\mathfrak{M}^n_q), \tag{3.11}$$

where $\tau' = (\tau')^\sharp \oplus (\tau')^\perp$ is the standard decomposition, and E is a non-negative sum of irreducible $\mathbb{H}_n$-modules which are not of the form $L_{\boldsymbol{\tau}}$ for any $\tau \in \mathcal{C}_{m_0}(\sigma)$.

Proof. Notice that $L^{\mathsf{A}}_{\boldsymbol{\tau}^\sharp} = M^{\mathsf{A}}_{\boldsymbol{\tau}^\sharp}$. By Corollary 2.8, every composition factor $L_{\boldsymbol{\tau}'}$ of $L^{\mathsf{A}}_{\boldsymbol{\tau}^\sharp} \circledast L_{\boldsymbol{\tau}^\perp}$ satisfies $\mathcal{O}_{\boldsymbol{\tau}} \subset \overline{\mathcal{O}_{\boldsymbol{\tau}'}}$. By Corollary 3.13, we deduce that $[M_{\boldsymbol{\tau}} : L_{\boldsymbol{\tau}'}] = 1$ for every $\tau' \in \mathcal{C}_{m_0}(\sigma)$ such that $\mathcal{O}_{\boldsymbol{\tau}} \subset \overline{\mathcal{O}_{\boldsymbol{\tau}'}}$. Consequently, we have

$$[L^{\mathsf{A}}_{\boldsymbol{\tau}^\sharp} \circledast L_{\boldsymbol{\tau}^\perp} : L_{\boldsymbol{\tau}'}] \le 1 \text{ for every } \tau' \in \mathcal{C}_{m_0}(\sigma).$$

By induction-by-stages, we have

$$L^{\mathsf{A}}_{\boldsymbol{\tau}^\sharp} \circledast M_{\boldsymbol{\tau}^\perp} \cong L^{\mathsf{A}}_{I_1} \circledast (L^{\mathsf{A}}_{I_2} \circledast (\cdots \circledast (L^{\mathsf{A}}_{I_N} \boxtimes M_{\boldsymbol{\tau}^\perp}))), \tag{3.12}$$

where $\boldsymbol{\tau}^\sharp = \{I_k\}_{k=1}^N$. If $N = 0$, there is nothing to prove. Let $n' := n - \#I_1$ and σ' be the Young diagram obtained by extracting a hook corresponding to I_1 from σ. Consider the following assertion:

($\heartsuit$) The $\mathbb{H}$-module $L^{\mathsf{A}}_{I_1} \circledast L_{\widetilde{\boldsymbol{\tau}}}$ contains both of $L_{\widetilde{\boldsymbol{\tau}}^{(1)}}$ and $L_{\widetilde{\boldsymbol{\tau}}^{(2)}}$ as composition factors for every $\widetilde{\tau} \in \mathcal{C}_{m_0}(\sigma')$, and $\widetilde{\tau}^{(1)}, \widetilde{\tau}^{(2)} \in \mathcal{C}_{m_0}(\sigma)$ that satisfy $\widetilde{\boldsymbol{\tau}}^{\sharp} = (\widetilde{\boldsymbol{\tau}}^{(1)})^{\sharp}$ and $\widetilde{\boldsymbol{\tau}}^{\sharp} \cup \{I_1\} = (\widetilde{\boldsymbol{\tau}}^{(2)})^{\sharp}$.

If ($\heartsuit$) holds, and (3.11) holds for all smaller N, then the comparison of multiplicity yields $[L^{\mathsf{A}}_{I_1} \circledast L_{\widetilde{\boldsymbol{\tau}}}] = E' + [L_{\widetilde{\boldsymbol{\tau}}^{(1)}}] + [L_{\widetilde{\boldsymbol{\tau}}^{(2)}}]$, where E' is a non-negative (formal) linear combination of irreducible $\mathbb{H}_n$-modules which are not isomorphic to $L_{(\mathsf{c}^m_\sigma, \tau)}$ for some $\tau \in \mathcal{C}_{m_0}(\sigma)$. Therefore, in order to prove (3.12), it suffices to verify ($\heartsuit$) provided that (3.11) holds for all smaller N cases.

Set $\widetilde{\boldsymbol{\tau}}^{+} := \{I_1\} \oplus \widetilde{\boldsymbol{\tau}}$. We have $\widetilde{\boldsymbol{\tau}}^{+} \in \mathcal{C}_{m_0}(\sigma)$. By Corollary 2.8, every composition factor $L_{\boldsymbol{\tau}'}$ of $(L^{\mathsf{A}}_{I_1} \circledast L_{\widetilde{\boldsymbol{\tau}}})$ satisfies $\mathcal{O}_{\widetilde{\boldsymbol{\tau}}^{+}} \subset \overline{\mathcal{O}_{\boldsymbol{\tau}'}}$. By Corollary 3.13, we deduce that $[M_{\widetilde{\boldsymbol{\tau}}^{+}} : L_{\boldsymbol{\tau}'}] = 1$ for every $\tau' \in \mathcal{C}_{m_0}(\sigma)$ such that $\mathcal{O}_{\widetilde{\boldsymbol{\tau}}^{+}} \subset \overline{\mathcal{O}_{\boldsymbol{\tau}'}}$. To show ($\heartsuit$), it suffices to verify that

$$[L^{\mathsf{A}}_{I_1} \circledast L_{\widetilde{\boldsymbol{\tau}}'} : L_{\widetilde{\boldsymbol{\tau}}^{(i)}}] = 0$$

for $i = 1, 2$ and every irreducible constituent $L_{\widetilde{\boldsymbol{\tau}}'} \not\cong L_{\widetilde{\boldsymbol{\tau}}}$ of $M_{\widetilde{\boldsymbol{\tau}}}$. This follows if

($\heartsuit$)$'$ $\overline{\mathcal{O}_{\widetilde{\boldsymbol{\tau}}^{(i)}}}$ does not contain $\mathcal{O}_{\{I_1\} \oplus \widetilde{\boldsymbol{\tau}}'}$

holds by Corollary 2.8. Here we have

$$\begin{aligned} &\dim \mathcal{O}_{\{I_1\} \oplus \widetilde{\boldsymbol{\tau}}'} > \dim \mathcal{O}_{\widetilde{\boldsymbol{\tau}}^{+}} \text{ and} \\ &\dim \mathcal{O}_{\widetilde{\boldsymbol{\tau}}^{(i)}} \le \dim \mathcal{O}_{\widetilde{\boldsymbol{\tau}}^{+}} + 1 \text{ (for each } i = 1, 2). \end{aligned} \tag{3.13}$$

Thus, in order to deduce inclusion, we have necessarily $\dim \mathcal{O}_{\{I_1\} \oplus \widetilde{\boldsymbol{\tau}}'} = \dim \mathcal{O}_{\widetilde{\boldsymbol{\tau}}^{+}} + 1$. It follows that $\widetilde{\boldsymbol{\tau}}'$ is obtained from $\widetilde{\boldsymbol{\tau}}$ by applying a unique elementary modification or putting one extra marking. Here I_1 corresponds to a hook of σ, but I_1 does not correspond to a hook of σ'. Therefore, we conclude that $\{I_1\} \oplus \widetilde{\boldsymbol{\tau}}' \neq \widetilde{\boldsymbol{\tau}}^{(i)}$ for $i = 1, 2$. This in turn implies ($\heartsuit$)$'$, and hence ($\heartsuit$). In conclusion, the induction proceeds and we obtain the result. $\square$

The rest of this section is devoted to the proof of Theorems 3.14, 3.15, and 3.16.

We apply Proposition 3.21 to τ_0 (borrowed from Lemma 3.20). Then, we obtain

$$[L^{\mathsf{A}}_{\boldsymbol{\tau}_0^{\sharp}} \circledast L_{\boldsymbol{\tau}_0^{\perp}}] = E + \sum_{\tau \in \mathcal{C}_{m_0}(\sigma)} [L_{\boldsymbol{\tau}}] \in K(\mathfrak{M}^n_{\vec{q}})$$

by Proposition 3.12. Here, $\lim_{m \to m_0} L^{\mathsf{A}}_{\boldsymbol{\tau}_0^{\sharp}}$ is a tempered module while $\lim_{m \to m_0} L_{\boldsymbol{\tau}_0^{\perp}}$ is a well-defined discrete series. It follows that every irreducible constituent of $L^{\mathsf{A}}_{\boldsymbol{\tau}_0^{\sharp}} \circledast L_{\boldsymbol{\tau}_0^{\perp}}$ is a tempered delimit. In particular, we have $\mathcal{C}_{m_0}(\sigma) \subset \mathcal{D}_{m_0}(\sigma)$. Moreover, Corollary 3.10 implies that $\mathcal{C}_{m_0}(\sigma) = \mathcal{D}_{m_0}(\sigma)$ by the comparison of the cardinality. This proves Theorem 3.14 and hence also Theorem 3.16. Moreover, we conclude $E = 0$ since there can be no other tempered delimits outside of $\mathcal{C}_{m_0}(\sigma)$. Therefore, we conclude Theorem 3.15 as desired.

3.4 Further properties of tempered delimits

We first recall a result based on the theory of analytic R-groups due, in the setting of affine Hecke algebras, to Delorme-Opdam [9]:

Theorem 3.22 (Slooten [30] Theorem 3.4.4)**.** *Let σ be a partition of n. Let $\mathbf{I} = \{I_1, \ldots, I_N\}$ be a multisegment consisting of segments with $\mathsf{E}_m(I_k) = 1$. Let d be the number of segments I_k of distinct size such that $e_+(I_k) \notin \{e_+(I) \mid I \in \mathsf{ds}_m(\sigma)\}$ and $e_-(I_k) \notin \{e_-(I) \mid I \in \mathsf{ds}_m(\sigma)\}$. Then, the module $L^{\mathsf{A}}_{\mathbf{I}} \circledast \mathsf{ds}_m(\sigma)$ is irreducible when $m \neq m_0$, and is a direct sum of 2^d irreducible components when $m = m_0$.*

Corollary 3.23. *For every $\tau \in \mathcal{D}_{m_0}(\sigma)$, the limit module $\lim_{m\to m_0} L_{\boldsymbol{\tau}}$ is irreducible.*

Proof. We borrow the notation $\boldsymbol{\tau}_0$ from Theorem 3.16. By Theorem 3.22, $\lim_{m\to m_0} L^{\mathsf{A}}_{\boldsymbol{\tau}_0^\sharp} \circledast L_{\boldsymbol{\tau}_0^\perp}$ splits into 2^h direct sums of tempered modules, where h is the number of segments in $\boldsymbol{\tau}_0^\sharp$. By Theorem 3.15, we know that $L^{\mathsf{A}}_{\boldsymbol{\tau}_0^\sharp} \circledast L_{\boldsymbol{\tau}_0^\perp}$ contains 2^h irreducible constituent even at generic m. It follows that all of such irreducible constituents, which is the whole of $\mathcal{D}_{m_0}(\sigma)$, must be irreducible by taking limit $m \to m_0$. $\square$

Lemma 3.24. *For every distinct choice of partitions σ, σ' of n, we have*

$$\mathcal{D}_{m_0}(\sigma) \cap \mathcal{D}_{m_0}(\sigma') = \emptyset.$$

Proof. Let $\chi \in \mathsf{P}(\vec{q})$ be a parameter corresponding to an element of $\mathcal{D}_{m_0}(\sigma)$. It is sufficient to prove that σ is canonically recovered from $\chi_0 := \lim_{m\to m_0} \chi^0 \in \mathsf{Q}(q)$.

Let $I \in \chi_0$ be a segment so that **1)** I or I^{-1} is of the form $\{q^{m_0}, q^{m_0+1}, \ldots\}$ or $\{\ldots, q^{m_0-1}, q^{m_0}\}$, and **2)** $\max(I \cup I^{-1})$ attains the maximum among all the segments in χ_0 which satisfy condition **1)**. Such a segment I must be unique (if it exist) since it gives the first segment in the smallest nested component with respect to $\Subset$ (via the ds_m-algorithm, see also Definition 3.8).

Let $\chi' = (\mathbf{I}', \delta)$ be the marked subpartition of χ so that $\mathbf{I}'$ is the collection of all segments I' such that $I \Subset I'$. By the ds_m-algorithm and Definition 3.8, we deduce that χ' forms a nested component of χ such that either both $\min I'$ and $\max I'$ or both $(\min I')^{-1}$ and $(\max I')^{-1}$ are the maximal/minimal values of a hook extracted from σ. Moreover, the marking of I' determines either I' or $(I')^{-1}$ must belong to χ, and consequently we obtain the shape of all the intermediate segments of the ds_m-algorithm step 2). Therefore, χ' is determined uniquely from χ_0.

In particular, we can assume $\chi' = \emptyset$. Then, according to the marking of I, all the segments of χ must be either uniformly marked (after changing the marking within the equivalence class if necessary) or uniformly unmarked. It follows that every $I' \in \chi$ must satisfy $\mathsf{E}_m(I') > 1$ or $\mathsf{E}_m(I') < 1$ uniformly. Assume that $\mathsf{E}_m(I') > 1$ for every $I' \in \chi$. Then, we arrange χ as

$$\max I = \max I_1 > \max I_2 > \cdots > \max I_\ell.$$

A segment I_k appears in the ds_m-algorithm step 2) if and only if $I = I_1$ or $\min I_k = q^{-1} \min I_l$ for some $l < k$. All the others are union of two segments appearing in the ds_m-algorithm step 2). This recovers the all segments appearing in the ds_m-algorithm step 2), and hence recovers σ uniquely. The other case is completely analogous, and hence the result follows. $\square$

Theorem 3.25. *Every tempered irreducible $\mathbb{H}_{n,m_0}$-module is obtained as*

$$\lim_{m\to m_0} L^{\mathsf{A}}_{\mathbf{I}} \circledast L_{\boldsymbol{\tau}}$$

for a unique balanced multisegment $\mathbf{I}$ (along m_0) and $\tau \in \mathcal{D}_{m_0}(\sigma)$ for a unique partition σ of some n'. In particular, such $\lim_{m\to m_0} L^{\mathsf{A}}_{\mathbf{I}} \circledast L_{\boldsymbol{\tau}}$ is an irreducible $\mathbb{H}_{n,m_0}$-module.

Proof. By the Evens-Langlands classification [10] (see also [7]), a tempered module is written as a quotient of a parabolic induction of the form $L^{\mathsf{A}}_{\mathbf{I}'} \circledast \mathsf{ds}$, where $\mathbf{I}'$ is a multisegment and $\mathsf{ds} = \lim_{m\to m_0} \mathsf{ds}_m(\sigma')$ is a discrete series obtained from a partition σ'. Since every discrete series is a tempered delimit, we can further assume that σ' does not contain a balanced hook as in Theorem 3.16. (c.f. Theorem 3.15)

We set $\mathbf{I}''$ to be the collection of all distinct segments of $\mathbf{I}'$ so that we have $e_+(I'') \notin \{e_+(I) \mid I \in \mathsf{ds}_m(\sigma')\}$ and $e_-(I'') \notin \{e_-(I) \mid I \in \mathsf{ds}_m(\sigma')\}$ for every $I'' \in \mathbf{I}''$. Let $\mathbf{I}$ be

the multisegment obtained from $\mathbf{I}'$ by removing segments in $\mathbf{I}''$. By the irreducibility of tempered induction of affine Hecke algebras of $GL(n)$, we deduce $L^{\mathsf{A}}_{\mathbf{I}} \circledast (L^{\mathsf{A}}_{\mathbf{I}''} \circledast \mathsf{ds}) \cong L^{\mathsf{A}}_{\mathbf{I}'} \circledast \mathsf{ds}$. Then, Theorem 3.22 implies that both $L^{\mathsf{A}}_{\mathbf{I}''} \circledast \mathsf{ds}$ and $L^{\mathsf{A}}_{\mathbf{I}'} \circledast \mathsf{ds}$ share the same number of irreducible constituents. It follows that $L^{\mathsf{A}}_{\mathbf{I}} \circledast L$ is irreducible for every irreducible constituent L of $L^{\mathsf{A}}_{\mathbf{I}'} \circledast \mathsf{ds}$. Here $\mathbf{I}' \oplus \mathsf{mp}_m(\sigma')$ is adapted to c^{σ}_m for the larger partition σ of n' by the construction of $\mathbf{I}'$. Therefore, we conclude $L = L_{\boldsymbol{\tau}}$ for some $\tau \in \mathcal{D}_{m_0}(\sigma)$, which implies the existence part of the assertion.

We prove the uniqueness of σ and $\tau \in \mathcal{D}_{m_0}(\sigma)$. Since the set $\mathbf{I}$ is uniquely determined by $\chi \in \mathsf{P}(\vec{q})$ corresponding to the tempered module, we can assume $\mathbf{I} = \emptyset$. Then, the assertion reduces to Lemma 3.24 as desired. □

Corollary 3.26 (of Corollary 3.13 and Theorem 3.15)**.** *Assume that some $\tau_{\max} \in \mathcal{D}_{m_0}(\sigma)$ satisfies $\mathsf{sgn} \subset L_{\boldsymbol{\tau}_{\max}}$ (as W-modules). Then, for every $\tau \in \mathcal{D}_{m_0}(\sigma)$, we have*

$$M_{\boldsymbol{\tau}} \cong L_{\boldsymbol{\tau}^{\sharp}} \circledast L_{\boldsymbol{\tau}^{\perp}}. \tag{3.14}$$

In addition, if $m_0 = 1/2$ or 1, then for each such τ, there exist

- *a nilpotent element $x = x_\tau \in \mathfrak{g}$ (with $\mathfrak{g} = \mathfrak{so}_{2n+1}$ and $\mathfrak{sp}_{2n}$, respectively);*
- *an irreducible representation $\xi = \xi_\tau$ of A_x appearing in the Springer correspondence, where $A_x = \mathsf{Stab}_G(x_\tau)/\mathsf{Stab}_G(x_\tau)^\circ$ (with $G = SO(2n+1)$ and $Sp(2n)$, respectively);*

such that we have

$$[H_\bullet(\mathcal{B}_x)_\xi] = \sum_{\tau' \in \mathcal{D}_{m_0}(\sigma); (\tau')^\sharp \subset \tau^\sharp} (-1)^{\#\boldsymbol{\tau}^\sharp - \#(\boldsymbol{\tau}')^\sharp} [H_\bullet(\mathcal{E}_{\boldsymbol{\tau}'})] \tag{3.15}$$

as virtual W-modules (without gradings). Here $\mathcal{B}_x$ is the Springer fiber of (the flag variety of G along) x and the subscript ξ means the ξ-isotypic component as A_x-modules.

Proof. Choose $m_0 < m < m_0 + \frac{1}{2}$. By [7] Theorem 1.22, our assumption is equivalent to the fact that $\mathcal{O}_{\boldsymbol{\tau}_{\max}}$ is the open dense orbit of $\mathbb{V}^{(\mathsf{c}^{\sigma}_m, \vec{q})}$. Taking into account Theorem 3.15, Corollary 3.13, and the Ginzburg theory (c.f. [12] Theorem 11.2), it suffices to prove that $\mathcal{O}_{\boldsymbol{\tau}}$ is not contained in any orbit closure except for $\mathcal{O}_{\boldsymbol{\tau}'}$ for some $\tau' \in \mathcal{D}_{m_0}(\sigma)$ in order to prove (3.14).

We prove this by contradiction. Let $\tau \in \mathcal{D}_{m_0}(\sigma)$ be a marked partition so that there exists a parameter χ which does not come from $\mathcal{D}_{m_0}(\sigma)$, but $\mathcal{O}_{\boldsymbol{\tau}} \subset \overline{\mathcal{O}_\chi}$. If $\tau^- \neq \emptyset$, then we deduce that $\tau_{\max}$ cannot define an open dense orbit. Thus, we assume $\tau^- = \emptyset$ in the following. Let $I_1, I_2, \ldots$ be the set of segments obtained from balanced hooks of σ along m_0. For each k, there exists at most one segment I of $\boldsymbol{\tau}$ such that $I_k \lhd I$; this is because $\mathcal{O}_{\boldsymbol{\tau}_{\max}}$ is open dense. Thus, we further conclude that $\varepsilon_{I_k, I}(\boldsymbol{\tau})$ comes from $\mathcal{D}_{m_0}(\sigma)$ for every $I \in \boldsymbol{\tau}$. Therefore, χ cannot exist, which finishes the first part of the proof.

For the latter part, notice that every tempered $\mathbb{H}_{n,m_0}$-module is of the form $H_\bullet(\mathcal{B}_x)_\xi$ by Kazhdan-Lusztig [14] Theorem 8.2 (before taking fixed points, but this does not affect the $\mathbb{H}^f_{n,m_0}$- or W-module structures). Therefore, inverting the multiplicity matrix (of simple modules in standard modules as $\mathbb{H}_{n,m}$-modules) given by (3.14) and Theorem 3.15 for $\mathcal{D}_{m_0}(\sigma)$ yields the assertion for m. By Corollary 3.23, the assertion holds for the limit $m \to m_0$ as required. □

Remark 3.27. (a) If $\xi = 1$, then the result becomes simpler as in [7] Corollary 1.23 (the right hand side of (3.15) is only $[H_\bullet(\mathcal{E}_{\tau_{\max}})]$).

(b) In view of Lusztig [22] Theorem 1.22, the same statement holds at all critical values $m_0 \geq 3/2$, but the homology of the classical Springer fiber in the left hand side of (3.15) is replaced by the appropriate homology group. It should be added that Corollary 3.26 becomes weaker for larger values of m_0, since it becomes more difficult for $\tau_{\max}$ with the desired property to exist.

3.5 An inductive algorithm for characters of tempered delimits

We give an inductive algorithm for computing W-characters of tempered delimits, and in particular of discrete series and limits of discrete series for all values of the parameter m. Fix a partition σ of n, and we retain the notation as before. In the following

$$\Theta_W(\pi) = \sum_{\chi \in \widehat{W}} [\pi : \chi]_{\mathbb{C}[W]} \, \chi$$

denotes the W-character of a module π. The induction proceeds in two ways: increasingly on the rank n, and decreasingly on the value m of the parameter.

Remark 3.28. Taking into account Theorem 3.25, we know that the W-character of every tempered module is obtained as an induced module of a suitable tempered delimits.

First, we fix a parameterization of $\widehat{W}_n$: Let $\mathbb{C}[\epsilon_1, \ldots, \epsilon_n]$ be the polynomial ring in which W_n acts naturally by extending the action on $R \subset X^*(T)$. For a bi-partition (μ, ν) of n, we define

$$\mathsf{p}_i^0(\mu) := \prod_{\mu_i^< < k < l \leq \mu_i^\leq} (\epsilon_k^2 - \epsilon_l^2), \; \mathsf{p}_i^+(\mu, \nu) := \prod_{\nu_i^< < k < l \leq \nu_i^\leq} (\epsilon_{k+|\mu|}^2 - \epsilon_{l+|\mu|}^2), \text{ and}$$
$$\mathsf{p}(\mu, \nu) := \prod_{i > |\mu|} \epsilon_i \times \prod_{i \geq 1} \mathsf{p}_i^0(\mu) \mathsf{p}_i^+(\mu, \nu).$$

Lemma 3.29 (cf. [5],§11.4)**.** *Let (μ, ν) be a bi-partition of n. Then the following W_n-module is irreducible:*

$$\{\mu, \nu\} := \mathbb{C}[W_n] \mathsf{p}({}^{\mathsf{t}}\mu, {}^{\mathsf{t}}\nu) \subset \mathbb{C}[\epsilon_1, \ldots, \epsilon_n].$$

Algorithm 3.30.

Step 0. Let $m_0 = n - 1$ be the critical point. If $m > n - 1$ then [7] gives that $\Theta_W(\mathsf{ds}_m(\sigma)) = \{\emptyset, {}^{\mathsf{t}}\sigma\}$.

Step 1. Consider $m_0 < m < m_0 + \frac{1}{2}$. Applying Theorem 3.15 in $\mathcal{D}_{m_0}(\sigma)$, we find that

$$\Theta_W(L^{\mathsf{A}}_{\boldsymbol{\tau}^{\mathsf{e}}_m} \circledast L_{\boldsymbol{\tau}^{\mathsf{s}}_m}) = \sum_{\tau' \in \mathcal{D}_{m_0}(\sigma); (\boldsymbol{\tau}')^{\mathsf{e}}_m \subset \boldsymbol{\tau}^{\mathsf{e}}_m} \Theta_W(L_{\boldsymbol{\tau}'_m}). \tag{3.16}$$

Solving this linear system gives

$$\Theta_W(L_{\boldsymbol{\tau}}) = \sum_{\tau' \in \mathcal{D}_{m_0}(\sigma); (\boldsymbol{\tau}')^{\mathsf{e}}_m \subset \boldsymbol{\tau}^{\mathsf{e}}_m} (-1)^{\#(\boldsymbol{\tau}^{\mathsf{e}}_m \setminus (\boldsymbol{\tau}')^{\mathsf{e}}_m)} \, \Theta_W(L^{\mathsf{A}}_{(\boldsymbol{\tau}')^{\mathsf{e}}_m} \circledast L_{(\boldsymbol{\tau}')^{\mathsf{s}}_m}), \quad \tau \in \mathcal{D}_{m_0}(\sigma) \tag{3.17}$$

Notice that in this equation, the right hand side is known since $(\boldsymbol{\tau}')^{\mathsf{s}} = \mathsf{ds}_m(\sigma')$ for some partition σ' of $n' \leq n$.

Step 2. Set $m = m_0$. By Corollary 3.23, for every $\tau \in \mathcal{D}_{m_0}(\sigma)$, the module $L(\tau) := \lim_{m\to m_0} L_{\boldsymbol{\tau}}$ is irreducible. Hence by Corollary 3.23 and by previous step, we get the W-character $\Theta_W(L(\tau))$ for every $\tau \in \mathcal{D}_{m_0}(\sigma)$.

Step 3. Consider $m_0 - \frac{1}{2} < m < m_0$. We need to find $\Theta_W(\mathsf{ds}_m(\sigma))$ (in order proceed with the algorithm). By Corollary 3.23, there exists a unique $\tau \in \mathcal{D}_{m_0}(\sigma)$, such that $\lim_{m\to m_0} \mathsf{ds}_m(\sigma) = L(\tau)$, and in particular, $\Theta_W(\mathsf{ds}_m(\sigma)) = \Theta_W(L(\tau))$. Applying Lemma 3.9, this τ is characterized by $\boldsymbol{\tau}^\sharp$, which is precisely the set of unmarked balanced segments of $\mathsf{ds}_m(\sigma)$.

Step 4. If $m_0 > 1 - n$, set $m_0 = m_0 - \frac{1}{2}$ and move to **Step 1**.

Remark 3.31. Notice that (3.17) remains valid if we replace Θ_W with ch. Moreover, we can compute $\mathsf{ch}(\mathsf{ds}_m(\sigma))$ for $m \gg 0$ (by [7] §4.7), and $\mathsf{ch}(L^{\mathsf{A}} \circledast L)$ from $\mathsf{ch}L^{\mathsf{A}}$ and $\mathsf{ch}L$. Since $\mathsf{ch}L_{\boldsymbol{\tau}}$ depends on m holomorphically in the region $m_0 - \frac{1}{2} < m < m_0 + \frac{1}{2}$ for each $\tau \in \mathcal{D}_{m_0}(\sigma)$, we can also compute $\mathsf{ch}L_{\boldsymbol{\tau}}$ by using the above algorithm.

Example 3.32. Consider the case $n = 6$ and $\sigma = (2, 2, 2)$, so that we have $\{\epsilon_i(\mathsf{c}^\sigma_m)\}_{i=1}^6 = \{q^{m+1}, q^m, q^m, q^{m-1}, q^{m-1}, q^{m-2}\}$. Then we find the following cases:

1. $m_0 < m < m_0 + \frac{1}{2}$ with $m_0 \in \frac{1}{2}\mathbb{Z}$ and $m_0 \geq 2$. We have $\mathcal{D}_{m_0}(\sigma) = \{\mathsf{ds}_m(\sigma)\}$, where $\Theta_W(\mathsf{ds}_m(\sigma)) = \{(0)(3^2)\}$.
2. $\frac{3}{2} < m < 2$. We have $\mathcal{D}_{\frac{3}{2}}(\sigma) = \{\mathsf{ds}_m(\sigma), L_{\tau^1_m}\}$, where:
   (a) $\Theta_W(\mathsf{ds}_m(\sigma)) = \{(0)(3^2)\}$;
   (b) $\Theta_W(L_{\tau^1_m}) = \{(0)(321)\}+\{(0)(2^21^2)\}+\{(1)(2^21)\}+\{(1)(32)\}+\{(1^2)(2^2)\}$ $((\boldsymbol{\tau}^1_m)^{\mathsf{e}} = \{q^{m-2}, q^{m-1}\})$.
3. $m = \frac{3}{2}$. $\mathcal{D}_{\frac{3}{2}}(\sigma)$ is as before, but $\lim_{m\searrow\frac{3}{2}} \mathsf{ds}_m(\sigma)$ is not a discrete series.
4. $1 < m < \frac{3}{2}$. We have $\mathcal{D}_1(\sigma) = \{\mathsf{ds}_m(\sigma), L_{\tau^1_m}, L_{\tau^2_m}, L_{\tau^3_m}\}$, where:
   (a) $\Theta_W(\mathsf{ds}_m(\sigma)) = \{(0)(321)\} + \{(0)(2^21^2)\} + \{(1)(2^21)\} + \{(1)(32)\} + \{(1^2)(2^2)\}$;
   (b) $\Theta_W(L_{\tau^1_m}) = \{(0)(31^3)\} + \{(0)(21^4)\} + \{(1)(31^2)\} + \{(1)(21^3)\} + \{(1^2)(31)\} + \{(1^2)(21^2)\} + \{(1^3)(21)\}$ $((\boldsymbol{\tau}^1_m)^{\mathsf{e}} = \{q^{m-2}, q^{m-1}\})$;
   (c) $\Theta_W(L_{\tau^2_m}) = \{(1)(2^21)\} + \{(2)(2^2)\} + \{(0)(2^3)\}$ $((\boldsymbol{\tau}^2_m)^{\mathsf{e}} = \{q^{m-1}\})$;
   (d) $\Theta_W(L_{\tau^3_m}) = \{(0)(1^6)\} + \{(0)(2, 1^4)\} + \{(0)(2^21^2)\} + 2\{(1)(1^5)\} + 2\{(1^2)(1^4)\} + 2\{(1^3)(1^3)\} + \{(1^4)(1^2)\} + 2\{(1)(21^3)\} + \{(2)(1^4)\} + 2\{(1^2)(21^2)\} + \{(2)(21^2)\} + \{(1)(2^21)\}+\{(21)(1^3)\}+\{(1^3)(21)\}+\{(21^2)(1^2)\}+\{(1^2)(2^2)\}+\{(21)(21)\}$ $((\boldsymbol{\tau}^3_m)^{\mathsf{e}} = \{(\boldsymbol{\tau}^1_m)^{\mathsf{e}}, (\boldsymbol{\tau}^2_m)^{\mathsf{e}}\})$.
5. $m = 1$. $\mathcal{D}_1(\sigma)$ is as before, but $\lim_{m\searrow 1} \mathsf{ds}_m(\sigma)$ is not a discrete series.
6. $\frac{1}{2} < m < 1$. We have $\mathcal{D}_{\frac{1}{2}}(\sigma) = \{\mathsf{ds}_m(\sigma), L_{\tau^1_m}, L_{\tau^2_m}, L_{\tau^3_m}\}$, where:
   (a) $\Theta_W(\mathsf{ds}_m(\sigma)) = \{(0)(1^6)\} + \{(0)(21^4)\} + \{(0)(2^21^2)\} + 2\{(1)(1^5)\} + 2\{(1^2)(1^4)\} + 2\{(1^3)(1^3)\} + \{(1^4)(1^2)\} + 2\{(1)(21^3)\} + \{(2)(1^4)\} + 2\{(1^2)(21^2)\} + \{(2)(21^2)\} + \{(1)(2^21)\} + \{(21)(1^3)\} + \{(1^3)(21)\} + \{(21^2)(1^2)\} + \{(1^2)(2^2)\} + \{(21)(21)\}$;
   (b) $\Theta_W(L_{\tau^1_m}) = \{(1^3)(21)\} + \{(21^2)(2)\} + \{(1^4)(2)\} + \{(1^4)(1, 1)\} + \{(21^3)(1)\} + \{(1^5)(1)\}$ $((\boldsymbol{\tau}^1_m)^{\mathsf{e}} = \{q^{m+1}, q^m, q^{m-2}, q^{m-2}\})$;
   (c) $\Theta_W(L_{\tau^2_m}) = \{(1^2)(1^4)\} + \{(21)(1^3)\} + \{(2^2)(1^2)\}$ $((\boldsymbol{\tau}^2_m)^{\mathsf{e}} = \{q^m, q^{m-1}\})$;

(d) $\Theta_W(L_{\tau^3_m}) = \{(1^3)(1^3)\} + \{(1^4)(1^2)\} + \{(21^2)(1^2)\} + \{(21^3)(1)\} + \{(1^5)(1)\} + \{(1^6)(0)\} + \{(2,1^4)(0)\} + \{(2^21^2)(0)\} + \{(2^21)(1)\}$ $((\boldsymbol{\tau}^3_m)^{\mathsf{e}} = \{(\boldsymbol{\tau}^1_m)^{\mathsf{e}}, (\boldsymbol{\tau}^2_m)^{\mathsf{e}}\})$.

7. $m = \frac{1}{2}$. $\mathcal{D}_{\frac{1}{2}}(\sigma)$ is as before, but $\lim_{m \searrow \frac{1}{2}} \mathsf{ds}_m(\sigma)$ is not a discrete series.

8. $0 < m < \frac{1}{2}$. We have $\mathcal{D}_0(\sigma) = \{\mathsf{ds}_m(\sigma), L_{\tau^1_m}\}$, where:

   (a) $\Theta_W(\mathsf{ds}_m(\sigma)) = \{(1^3)(1^3)\} + \{(1^4)(1^2)\} + \{(21^2)(1^2)\} + \{(21^3)(1)\} + \{(1^5)(1)\} + \{(1^6)(0)\} + \{(21^4)(0)\} + \{(2^21^2)(0)\} + \{(2^21)(1)\}$;

   (b) $\Theta_W(L_{\tau^1_m}) = \{(2^3)(0)\}$ $((\boldsymbol{\tau}^1_m)^{\mathsf{e}} = \{q^{m+1}, q^m, q^{m-1}\})$.

9. $m = 0$. $\mathcal{D}_0(\sigma)$ is as before, but $\lim_{m \searrow 0} \mathsf{ds}_m(\sigma)$ is not a discrete series.

10. $m_0 < m < m_0 + \frac{1}{2}$ with $m_0 \in \frac{1}{2}\mathbb{Z}_{<0}$. We have $\mathcal{D}_{m_0}(\sigma) = \{\mathsf{ds}_m(\sigma)\}$, where $\Theta_W(\mathsf{ds}_m(\sigma)) = \{(2^3)(0)\}$.

□

We explain that the algorithm gives also the W_n^{D}-character of discrete series for $\mathbb{H}_{n,m}^{\mathsf{D}}$. For every partition σ of n, we set $L_0(\sigma) := \lim_{m \to 0} \mathsf{ds}_m(\sigma)$.

Lemma 3.33 (cf. [5],§11.4)**.** *Recall the W_n-representation $\{\mu, \nu\}$ from Lemma 3.29.*

1. *The restriction of $\{\mu, \nu\}$ to W_n^{D} is irreducible unless $\mu = \nu$;*
2. *We have $\{\mu, \nu\} \cong \{\nu, \mu\}$ as W_n^{D}-modules;*
3. *We have $\{\mu, \nu\} \otimes \mathsf{sgn} \cong \{{}^{\mathsf{t}}\nu, {}^{\mathsf{t}}\mu\}$;*
4. *The dimension of $\mathrm{Hom}_{W_n^{\mathsf{D}}}(\{\mu, \nu\}, \{\mu', \nu'\})$ equals:*

$$\begin{cases} 1, & \text{if } (\mu,\nu) \in \{(\mu',\nu'),(\nu',\mu')\}, \text{ but } \mu \neq \nu, \\ 4, & \text{if } (\mu,\nu) \in \{(\mu',\nu'),(\nu',\mu')\}, \text{ and } \mu = \nu, \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 3.34. *The restriction of $L_0(\sigma)$ from $\mathbb{H}_{n,0}$ to $\mathbb{H}_n^{\mathsf{D}}$ is irreducible.*

Proof. In this proof, we freely identify $\mathbb{H}_n^f$ with $\mathbb{C}[W_n]$, and $\mathbb{H}_n^{\mathsf{D},f}$ with $\mathbb{C}[W_n^{\mathsf{D}}]$. Applying the ds_m-algorithm to σ ($0 < m < \frac{1}{2}$), we deduce that the largest segment $I \in \mathsf{ds}_m(\sigma)$ (with respect to the cardinality) satisfies $q^m \in I$. Let (μ, ν) be the bi-partition corresponding to $G\mathcal{O}_{\mathsf{ds}_m(\sigma)}$ by [13] Theorem 5.1. We have $L_\sigma := \{{}^{\mathsf{t}}\nu, {}^{\mathsf{t}}\mu\} \subset L_0(\sigma)$ by the exotic Springer correspondence. If $\mathsf{E}_m(I) < 1$, then we deduce that $\mu_1 < \nu_1$ from the fact that I is not marked. If $\mathsf{E}_m(I) > 1$, then we have $\mu_1 > \nu_1$. Thanks to Lemma 3.33 and [13] Theorem 10.7, we conclude that the restriction of L_σ from W_n to W_n^{D} is irreducible. (Notice that the correspondences in [12] and [13] are equivalent under tensoring with sgn, and the correspondences in [13] and Lemma 3.33 are equivalent, respectively.)

By Lemma 3.33, we have $[L_0(\sigma) : L_\sigma]_{\mathbb{C}[W_n^{\mathsf{D}}]} > 1$ only when $[L_0(\sigma) : L'_\sigma]_{\mathbb{C}[W_n]} > 0$ with $L'_\sigma := \{{}^{\mathsf{t}}\mu, {}^{\mathsf{t}}\nu\}$. In particular, there exists a marked partition τ corresponding to L'_σ and $\mathcal{O}_{\mathsf{ds}_m(\sigma)} \subset G\mathcal{O}_{\mathsf{ds}_m(\sigma)} \subset \overline{\mathcal{O}_\tau}$. This happens only if $\mathsf{E}_m(I) < 1$ since we need $\mu_1 \le \nu_1$ by [1]. Moreover, we have $\mathsf{pr}(G\mathcal{O}_{\mathsf{ds}_m(\sigma)}) = \mathsf{pr}(\mathcal{O}_\tau)$ as G-orbits in $\mathbb{V}$, where pr is the projection $\mathbb{V}_n \to V_n^{(2)}$. We define Ξ to be the set of parameters χ which satisfy $\mathcal{O}_{\mathsf{ds}_m(\sigma)} \subset \overline{\mathcal{O}_\chi}$, $\mathsf{pr}(\mathcal{O}_{\mathsf{ds}_m(\sigma)}) = \mathsf{pr}(\mathcal{O}_\chi)$, and $G\mathcal{O}_\chi \subset \overline{\mathcal{O}_\tau}$. Recall that L_χ denotes the irreducible $\mathbb{H}_{n,m}$-module parameterized by χ.

Claim B. *If $[L_\chi : L'_\sigma]_{\mathbb{C}[W_n]} \neq 0$ for some $\chi \in \Xi$, then χ must be maximal with respect to the closure ordering in Ξ.*

Proof. The variety $\mathsf{pr}^{-1}(\mathsf{pr}(\mathcal{O}_{\mathsf{ds}_m(\sigma)})) \cap \mathbb{V}_n^{(\mathsf{c}_m^\sigma, \vec{q})}$ is a vector bundle over $\mathcal{O}_{\mathsf{ds}_m(\sigma)}$. Hence, a closure relation in Ξ just represents a vector subbundle over $\mathsf{pr}(\mathcal{O}_{\mathsf{ds}_m(\sigma)})$; this means that $\overline{\mathcal{O}_\chi}$ is smooth along $\mathcal{O}_{\chi'}$ for every $\chi, \chi' \in \Xi$ such that $\mathcal{O}_{\chi'} \subset \overline{\mathcal{O}_\chi}$. From this, we deduce $[M_{\mathsf{ds}_m(\sigma)} : L_\chi] = 1$ for $\chi \in \Xi$. By the analogous vector bundle structure for all of $\mathbb{V}_n$, we conclude $[M_\chi : L'_\sigma]_{\mathbb{C}[W_n]} = 1$. Applying the Ginzburg theory, we conclude that $[L_\chi : L'_\sigma]_{\mathbb{C}[W_n]} = 1$ for a unique $\chi \in \Xi$. In view of the above multiplicity estimates, we conclude that χ is maximal in Ξ with respect to the closure ordering. □

We return to the proof of Proposition 3.34.

In the case $\mathsf{E}_m(I) < 1$, we have $G\mathcal{O}_{\mathsf{ds}_m(\sigma)} \subset \overline{\mathcal{O}_{\chi'}} \subset \overline{\mathcal{O}_\chi}$ for a parameter χ' obtained from $\mathsf{ds}_m(\sigma) = (\mathsf{c}_m^\sigma, \tau)$ by adding an extra marking on $J \in \tau$ with $I = \underline{J}$. This implies that $\mathsf{ds}_m(\sigma)$ is not maximal in Ξ. Thus, we have necessarily $[L_0(\sigma) : L'_\sigma]_{\mathbb{C}[W_n]} = 0$. Therefore, we conclude $[L_0(\sigma) : L_\sigma]_{\mathbb{C}[W_n^{\mathsf{D}}]} = 1$.

We now show that $L_0(\sigma)$ is irreducible as an $\mathbb{H}_n^{\mathsf{D}}$-module. In order to deduce this by contradiction, we assume that there exists a proper $\mathbb{H}_n^{\mathsf{D}}$-submodule $M \subset L_0(\sigma)$. We have $\mathbb{H}_n^{\mathsf{D}} L_\sigma = \mathbb{H}_n^{\mathsf{D}} N_n L_\sigma = L_0(\sigma)$ since $\mathbb{H}_{n,0} = \mathbb{H}_n^{\mathsf{D}} + \mathbb{H}_n^{\mathsf{D}} N_n$. Because $[L_0(\sigma) : L_\sigma]_{\mathbb{C}[W_n^{\mathsf{D}}]} = 1$, we have $[M, L_\sigma]_{\mathbb{C}[W_n^{\mathsf{D}}]} = 0$. Since $L_0(\sigma)$ is irreducible as an $\mathbb{H}_{n,0}$-module, we have a surjection

$$\mathrm{Ind}_{\mathbb{H}_n^{\mathsf{D}}}^{\mathbb{H}_{n,0}} M = \mathbb{H}_{n,0} \otimes_{\mathbb{H}_n^{\mathsf{D}}} M \twoheadrightarrow L_0(\sigma).$$

As W_n^{D}-modules, $\mathrm{Ind}_{\mathbb{H}_n^{\mathsf{D}}}^{\mathbb{H}_{n,0}} M = M \oplus N_n M$, and therefore $N_n M$ contains L_σ as a W_n^{D}-module. However, L_σ must give rise to an irreducible W_n-submodule of $L_0(\sigma)$. Therefore, we conclude $L_\sigma \subset N_n M \cap M \subset M$. This is a contradiction and thus $L_0(\sigma)$ is irreducible as an $\mathbb{H}_n^{\mathsf{D}}$-module. □

4 Formal degrees

4.1 Preliminaries

In this section, we consider the Hecke algebra with three parameters $\mathbb{H}_n = \mathbb{H}_n(q, u, v)$, and assume that u and v are specialized to $u = q^{m_+}$ and $v = q^{m_-}$, where $q > 1$ and $m_\pm \in \mathbb{R}$. We retain the notation from §2.2. If $w \in \widetilde{W}_n$ has a reduced expression $w = s_{i_1} \cdot \cdots \cdot s_{i_k}$, $i_l \in \{0, 1, \ldots, n\}$, in terms of the affine simple reflections, then define the elements of $\mathbb{H}_n$, $N_w = N_{i_1} \cdot \cdots \cdot N_{i_k}$, where N_{i_l} are the generators of $\mathbb{H}_n$ from §2.2. This definition does not depend on the choice of reduced expression.

The algebra $\mathbb{H}_n$ has a structure of normalized Hilbert algebra (in the sense of [8] A.54), with the $*$ operation given on generators by

$$N_w^* = N_{w^{-1}}, \quad w \in \widetilde{W}_n,$$

the trace functional τ given by

$$\tau(N_w) = 0, \text{ if } w \neq 1, \text{ and } \tau(1) = 1,$$

and the inner product $[\ ,\ ]$ given by

$$[x, y] = \tau(x^* y), \text{ for all } x, y \in \mathbb{H}_n.$$

Since all of the irreducible $\mathbb{H}_n$-modules are finite dimensional, the trace tr is well defined on every irreducible module, and there exists a positive Borel measure $\hat{\mu}$ on the tempered dual $\widehat{\mathfrak{S}}$ of $\mathbb{H}_n$ such that the abstract Plancherel formula holds:

$$[x,1] = \int_{\widehat{\mathfrak{S}}} \mathsf{tr}\pi(x)\; d\hat{\mu}(\pi), \quad x \in \mathbb{H}. \tag{4.1}$$

Moreover, an irreducible tempered representation π has positive volume $\hat{\mu}(\pi) > 0$ if and only if π is a discrete series. In this case, we denote by $\mathrm{fd}(\pi) = \hat{\mu}(\pi)$ the formal degree of π. The formal degree $\mathrm{fd}(\pi)$ is known up to a rational constant C_π independent of q (but depending on π). The purpose of this section is to calculate this constant.

To begin, we have the following known result. Recall that R_n denotes the set of roots of T_n in $G_n = Sp(2n,\mathbb{C})$, and let us denote by R_n^{sh} and R_n^{lo} the short and the long roots, respectively.

Theorem 4.1 ([25], Theorem 4.6)**.** *If π is a discrete series of $\mathbb{H}_n$ with central character $s \in T_n$, there exists a rational constant C_π independent of q such that*

$$\mathrm{fd}(\pi) = \frac{C_\pi\; q^{n^2-n} q^{nm_+} \prod'_{\alpha\in R_n}(\alpha(s)-1)}{\prod'_{\alpha\in R_n^{\mathsf{sh}}}(q\alpha(s)-1)\prod'_{\alpha\in R_n^{\mathsf{lo}}}(q^{\frac{m_++m_-}{2}}\alpha(s)^{1/2}-1)\prod'_{\alpha\in R_n^{\mathsf{lo}}}(q^{\frac{m_+-m_-}{2}}\alpha(s)^{1/2}+1)}, \tag{4.2}$$

where $\prod'$ means that the product is taken only over the nonzero factors.

Notice also that the known part of (4.2) only depends on the central character $W_n s$, and not on π. Our strategy in the determination of the constant in the formula (4.2) is to compare this formula to an Euler-Poincaré type of formula which also gives the formal degree.

The second idea we need, that of the Euler-Poicaré function, traces back to Kottwitz and [29]. For the setting of the affine Hecke algebra, the reference is [24].

Convention 4.2. If S is a subset of $\widetilde{\Pi}_n$, let $\mathbb{H}_S^f$ and W_S denote the finite Hecke subalgebra of $\mathbb{H}_n$ and the finite subgroup of $\widetilde{W}_n$ generated by the roots in S, respectively. By Tits' deformation theorem, we have an isomorphism of algebras $\mathbb{H}_S^f \cong \mathbb{C}[W_S]$. We use this identification in the following, and for example, for every $\sigma \in \widehat{W_S}$, we denote by $\mathrm{gd}(\sigma)$ the generic degree of the corresponding $\mathbb{H}_S^f$-module. There exists an explicit formula for computing $\mathrm{gd}(\sigma)$ (see [5], page 447), which we recall later in (4.20).

Definition 4.3 ([24], equations (3.15),(3.19))**.** If π is a finite dimensional $\mathbb{H}_n$-module, define the Euler-Poincaré element f_π as follows:

$$f_\pi = \frac{1}{2}\sum_{S\subsetneq\widetilde{\Pi}_n}(-1)^{n-|S|}\sum_{\gamma\in\mathrm{Irr}(\mathbb{H}_S^f)}\frac{[\pi:\gamma]_{\mathbb{H}_S^f}\,\mathrm{gd}(\gamma)}{\dim\gamma}e_\gamma, \tag{4.3}$$

where $e_\gamma \in \mathbb{H}_S^f$ is the primitive central idempotent in corresponding to γ. For π,π' finite dimensional $\mathbb{H}_n$-modules, define the Euler characteristic

$$\mathrm{EP}(\pi,\pi') = \sum_{i\geq 0}(-1)^i \dim \mathrm{Ext}^i_{\mathbb{H}_n}(\pi,\pi'). \tag{4.4}$$

The remarkable property of f_π, established in this setting in [24] Proposition 3.6, is that one has

$$\mathsf{tr}\pi'(f_\pi) = \mathrm{EP}(\pi, \pi'), \tag{4.5}$$

for all finite dimensional $\mathbb{H}_n$-modules π, π'. Moreover, it is shown in [24] Theorem 3.8, that

$$\mathrm{Ext}^i_{\mathbb{H}_n}(\pi, \pi') = \mathbb{C}, \text{ if } \pi \cong \pi' \text{ and } i = 0, \quad \mathrm{Ext}^i_{\mathbb{H}_n}(\pi, \pi') = 0, \text{ otherwise}, \tag{4.6}$$

whenever π is an irreducible discrete series and π' is an irreducible tempered module. This allows one to use f_π in (4.1) to find the following formula for $\mathrm{fd}(\pi)$.

Theorem 4.4 ([24], Proposition 3.4)**.** *1. Let π be a discrete series $\mathbb{H}_n$-module. The formal degree of π is*

$$\mathrm{fd}(\pi) = [f_\pi, 1] = \frac{1}{2} \sum_{S \subsetneq \widetilde{\Pi}_n} (-1)^{n-|S|} \sum_{\gamma \in \widehat{W}_S} \frac{[\pi : \gamma]_{\mathbb{C}[W_S]} \mathrm{gd}(\gamma)}{P_S}, \tag{4.7}$$

where P_S is the Poincaré polynomial for the Hecke algebra $\mathbb{H}^f_S$.

2. Assume that π is irreducible tempered, but not a discrete series, or else, that it is a parabolically induced module from a discrete series on a proper Hecke subalgebra. Then, we have

$$[f_\pi, 1] = 0. \tag{4.8}$$

One can simplify formula (4.7), as in [26]. In the following, using Convention 4.2, we identify $\mathbb{C}[W_i \times W_{n-i}]$ with the corresponding finite Hecke subalgebra of $\mathbb{H}_n$.

Corollary 4.5. *Let π be a finite dimensional $\mathbb{H}_n$-module. Then one has*

$$[f_\pi, 1] = \frac{1}{2} \sum_{i=0}^{n} (-1)^{n-i} \sum_{\gamma_1 \boxtimes \gamma_2 \in \widehat{W_i \times W_{n-i}}} \frac{[\pi : \gamma_1 \boxtimes \gamma_2]_{\mathbb{C}[W_i \times W_{n-i}]} \mathrm{gd}(\gamma_1) \mathrm{gd}(\gamma_2 \otimes \mathsf{sgn})}{P_i(q, q^{m_-}) P_{n-i}(q, q^{m_+})}, \tag{4.9}$$

where $W_i \times W_{n-i}$ is the Coxeter group generated by the reflections in the roots of $\widetilde{\Pi}_n$ except the i-th root (the roots are numbered $0, \dots, n$), and P_j denotes the Poincaré polynomial for type B_j and corresponding labels.

Every quantity in formula (4.9) is computable, provided that we know the restrictions of π to $\mathbb{C}[W_i \times W_{n-i}] \subset \mathbb{H}_n$, for every $0 \leq i \leq n$.

4.2 The constant in formal degrees

Definition 4.6. A label (m_+, m_-), for the Hecke algebra $\mathbb{H}_n(q, q^{m_+}, q^{m_-})$ is called generic if $|m_+ \pm m_-| \notin \{0, 1, 2, \dots, 2n-1\}$, and it is called critical otherwise.

Let us recall briefly recall the part of the reduction to positive real central character for $\mathbb{H}_n$ ([20]) that is relevant to us. Assume π is a discrete series of $\mathbb{H}_n$ with central character $\mathsf{c}(\pi) \in T_n$, not necessarily positive. Then there exists k, $1 \leq k \leq n$, and two discrete series π_1, π_2 of $\mathbb{H}_{k,m_1}$ and $\mathbb{H}_{n-k,m_2}$, respectively, where $2|m_1| = m_+ - m_-$, $2|m_2| = m_+ + m_-$, such that π_1, π_2 have positive real central characters $\mathsf{c}(\pi_1)$, $\mathsf{c}(\pi_2)$, with $\mathsf{c}(\pi_1) \times \mathsf{c}(\pi_2) \in T_k \times T_{n-k} = T_n$, and

$$\begin{aligned}
&\mathsf{c}(\pi) = (-\mathsf{c}(\pi_1), \mathsf{c}(\pi_2)), \\
&\lim_{q \to 1} \pi = \mathrm{Ind}^{\widetilde{W}_n}_{\widetilde{W}_k \times \widetilde{W}_{n-k}} (\lim_{q \to 1} \pi_1 \boxtimes \lim_{q \to 1} \pi_2), \text{ and, in particular} \\
&\pi|_{W_n} = \mathrm{Ind}^{W_n}_{W_k \times W_{n-k}} (\pi_1|_{W_k} \boxtimes \pi_2|_{W_{n-k}});
\end{aligned} \tag{4.10}$$

here $\pi_1|_{W_k}$ and $\pi_2|_{W_{n-k}}$ are understood as restrictions in the Hecke algebra $\mathbb{H}_{k,m_1}$ and $\mathbb{H}_{n-k,m_2}$, respectively, and W_k, W_{n-k} in the induction are viewed as the subgroups of W_n generated by the reflections in the roots $\{\epsilon_1-\epsilon_2,\dots,\epsilon_{k-1}-\epsilon_k, 2\epsilon_k\}$ and $\{\epsilon_{k+1}-\epsilon_{k+2},\dots,\epsilon_{n-1}-\epsilon_n, 2\epsilon_n\}$, respectively. Here we remark that Algorithm 3.30 (and in particular Corollary 3.19) holds verbatim with respect to m_1 (for π_1) and m_2 (for π_2) independently.

We state the main result of this section.

Theorem 4.7. *Let π be a discrete series for $\mathbb{H}_n$ with central character $\mathsf{c}(\pi)$. If the parameters (m_+, m_-) for $\mathbb{H}_n$ are generic, then the constant in formula (4.2) for the formal degree* $\mathrm{fd}(\pi)$ *is (up to a sign) $C_\pi = \frac{1}{2}$.*

The proof is presented in the following subsections. The general case follows immediately from Theorem 4.7.

Corollary 4.8. *Let π be a discrete series for $\mathbb{H}_n$ with central character $\mathsf{c}(\pi)$. Then in formula (4.2), we have $C_\pi = \pm\frac{1}{2}$.*

Proof. Assume that π^0 is a discrete series for $\mathbb{H}_n$ for critical values (m_+^0, m_-^0) of the labels, and having central character $\mathsf{c}(\pi^0)$. Let k, $m_1^0 = \frac{m_+^0 - m_-^0}{2}$, $m_2^0 = \frac{m_+^0 + m_-^0}{2}$, and discrete series π_1^0, π_2^0 of $\mathbb{H}_{k,m_1^0}$ and $\mathbb{H}_{k,m_2^0}$ be as in the discussion around (4.10). Recall from §3 that π_1^0 and π_2^0 belong to families of discrete series indexed by partitions σ_1 of k and σ_2 of $n-k$, respectively. To emphasize this dependence, we write, as in §3, $\pi_i^0 = \mathsf{ds}_{m_i^0}(\sigma_i) = (\mathsf{c}_{m_i^0}^{\sigma_i}, ds_{m_i^0}(\sigma_i))$, $i = 1, 2$, where $\mathsf{c}_{m_i^0}^{\sigma_i}$ is the central character, and $ds_{m_i^0}(\sigma_i)$ is the marked partition adapted to $\mathsf{c}_{m_i^0}^{\sigma_i}$ that parameterizes the discrete series.

Set $m_i^t = m_i^0 + t$ for $t \in (-\frac{1}{2}, \frac{1}{2})$ and $i = 1, 2$, and consider the corresponding discrete series $\pi_i^t = \mathsf{ds}_{m_i^t}(\sigma_i) = (\mathsf{c}_{m_i^t}^{\sigma_i}, ds_{m_i^t}(\sigma_i))$. Then, by Corollary 3.23, we have

$$\lim_{t\to 0} \pi_i^t = \pi_i^0, \quad i = 1, 2. \tag{4.11}$$

For every $t \in (-\frac{1}{2}, \frac{1}{2})$, set $m_+^t = m_1^t + m_2^t$ and $m_-^t = m_2^t - m_1^t = m_{0,-}$, and let π^t be the discrete series module of $\mathbb{H}_n(q, q^{m_+^t}, q^{m_-^t})$ corresponding in the reduction to positive real central character to the pair (π_1^t, π_2^t). Notice that the labels (m_+^t, m_-^t) are generic, in the sense of Definition 4.6, for all $t \in (-\frac{1}{2}, \frac{1}{2}) \setminus \{0\}$. Moreover (4.11) implies that $\lim_{t\to 0} \pi^t = \pi^0$. This, in conjunction with Theorem 4.4 1), gives

$$\lim_{t\to 0} \mathrm{fd}(\pi^t) = \mathrm{fd}(\pi^0). \tag{4.12}$$

Since we assumed that π is a discrete series, this limit must be nonzero. Therefore, one only needs to analyze the factors in (4.2) for π^t that vanish at $t \to 0$. More precisely, with the notation

$$R_n(j) := \{\alpha \in R_n :\ \alpha(\mathsf{c}(\pi^0)) = q^j\}, \quad R_n^{\mathsf{lo}}(j)_\pm := \{\alpha \in R_n^{\mathsf{lo}} :\ \alpha(\mathsf{c}(\pi^0))^{1/2} = \pm q^j\},$$

we have:

$$\begin{aligned} C_{\pi^0} = \lim_{t\to 0} & \frac{C_{\pi^t} \prod'_{\alpha\in R_n(0)} (\alpha(\mathsf{c}(\pi^t)) - 1)}{\prod'_{\alpha\in R_n(-1)\cap R_n^{\mathsf{sh}}} (q\alpha(\mathsf{c}(\pi^t)) - 1) \prod'_{\alpha\in R_n^{\mathsf{lo}}(-m_2^0)_+} (q^{m_2^t}\alpha(\mathsf{c}(\pi^t))^{1/2} - 1)} \\ & \cdot \frac{1}{\prod'_{\alpha\in R_n^{\mathsf{lo}}(-m_1^0)_-} (q^{m_1^t}\alpha(\mathsf{c}(\pi^t))^{1/2} + 1)}. \end{aligned} \tag{4.13}$$

Firstly, notice that for all roots of the form $\alpha = \pm\epsilon_i \pm \epsilon_j$ such that $1 \le i \le k$, $k+1 \le j \le n$, we have $\alpha(\mathsf{c}(\pi^t)) < 0$ and therefore, these roots do not appear in (4.13). Secondly, we remark that for all the roots α (both short and long) that appear in (4.13), the corresponding factors must be of the form $\pm(q^{\pm 2t} - 1)$, where $t \to 0$. This implies that we have $C_{\pi^0} = \lim_{t \to 0} C_{\pi^t}$, and this proves the claim. $\square$

Before presenting the proof of Theorem 4.7, we explain how the result in Corollary 4.8 relates to the expected form of the formal degree in the case of affine Hecke algebras of $\mathbb{H}_{n,m}$, $\mathbb{H}'_{n,m}$, $\mathbb{H}^{\mathsf{D}}_n$ of types $\mathsf{C}_n, \mathsf{B}_n, \mathsf{D}_n$, respectively. To emphasize the type of the root system, let $R^{\mathsf{C}}_n, R^{\mathsf{B}}_n, R^{\mathsf{D}}_n$, denote the roots in these cases. We have $R^{\mathsf{C}}_n = R_n$, $R^{\mathsf{B}}_n = \check{R}_n$, and $R^{\mathsf{D}}_n = R^{\mathsf{sh}}_n$.

(1) For $\mathbb{H}_{n,m}$, we specialize $m_+ = m$, $m_- = m$. Assuming π^{C} is a discrete series with central character s, we find:

$$\begin{aligned} \mathrm{fd}(\pi^{\mathsf{C}}) &= \frac{q^{n^2-n} q^{nm} \prod'_{\alpha \in R_n} (\alpha(s) - 1)}{2 \prod'_{\alpha \in R^{\mathsf{sh}}_n} (q\alpha(s) - 1) \prod'_{\alpha \in R^{\mathsf{lo}}_n} (q^m \alpha(s)^{1/2} - 1) \prod'_{\alpha \in R^{\mathsf{lo}}_n} (\alpha(s)^{1/2} + 1)} \\ &= C^{\mathsf{C}}_\pi \, \frac{q^{n^2-n} q^{nm} \prod'_{\check\alpha \in \check{R}^{\mathsf{C}}_n} (\check\alpha(s) - 1)}{\prod'_{\check\alpha \in \check{R}^{\mathsf{C}}_n} (q(\alpha)\check\alpha(s) - 1)}, \quad q(\alpha) = \begin{cases} q, & \alpha \text{ short} \\ q^m, & \alpha \text{ long} \end{cases}, \end{aligned} \tag{4.14}$$

where

$$C^{\mathsf{C}}_\pi = \frac{1}{2^{e^{\mathsf{C}}+1}}, \quad e^{\mathsf{C}} = 2\#\{i : 1 \le i \le n,\ \epsilon_i(s) = 1\}. \tag{4.15}$$

(2) For $\mathbb{H}'_{n,m}$, we specialize $m_+ = 2m$, $m_- = 0$. Assuming π^{B} is a discrete series with central character s, we find:

$$\begin{aligned} \mathrm{fd}(\pi^{\mathsf{B}}) &= \frac{q^{n^2-n} q^{2nm} \prod'_{\alpha \in R_n} (\alpha(s) - 1)}{\prod'_{\alpha \in R^{\mathsf{sh}}_n} (q\alpha(s) - 1) \prod'_{\alpha \in R^{\mathsf{lo}}_n} (q^m \alpha(s)^{1/2} - 1) \prod'_{\alpha \in R^{\mathsf{lo}}_n} (q^m \alpha(s)^{1/2} + 1)} \\ &= C^{\mathsf{B}}_\pi \, \frac{q^{n^2-n} q^{2nm} \prod'_{\check\alpha \in \check{R}^{\mathsf{C}}_n} (\check\alpha(s) - 1)}{\prod'_{\check\alpha \in \check{R}^{\mathsf{C}}_n} (q(\alpha)\check\alpha(s) - 1)}, \quad q(\alpha) = \begin{cases} q, & \alpha \text{ short} \\ q^{2m}, & \alpha \text{ long} \end{cases}, \end{aligned} \tag{4.16}$$

where

$$C^{\mathsf{B}}_\pi = \frac{1}{2^{e^{\mathsf{B}}}}, \quad e^{\mathsf{B}} = \#\{i : 1 \le i \le n,\ \epsilon_i(s) = \pm q^m\} + \#\{i : 1 \le i \le n,\ \epsilon_i(s) = \pm q^{-m}\}. \tag{4.17}$$

(3) For $\mathbb{H}^{\mathsf{D}}_n$, we specialize $m_+ = 0$, $m_- = 0$. Assuming π^{D} is a discrete series with central character s, we find:

$$\begin{aligned} \mathrm{fd}(\pi^{\mathsf{D}}) &= \frac{q^{n^2-n} \prod'_{\alpha \in R_n} (\alpha(s) - 1)}{\prod'_{\alpha \in R^{\mathsf{sh}}_n} (q\alpha(s) - 1) \prod'_{\alpha \in R^{\mathsf{lo}}_n} (\alpha(s)^{1/2} - 1) \prod'_{\alpha \in R^{\mathsf{lo}}_n} (\alpha(s)^{1/2} + 1)} \\ &= C^{\mathsf{D}}_\pi \, \frac{q^{n^2-n} \prod'_{\check\alpha \in \check{R}^{\mathsf{D}}_n} (\check\alpha(s) - 1)}{\prod'_{\check\alpha \in \check{R}^{\mathsf{D}}_n} (q\check\alpha(s) - 1)}, \end{aligned} \tag{4.18}$$

where

$$C^{\mathsf{D}}_\pi = \frac{1}{2^{e^{\mathsf{D}}}}, \quad e^{\mathsf{D}} = 2\#\{i : 1 \le i \le n,\ \epsilon_i(s) = \pm 1\}. \tag{4.19}$$

In types B and D, we needed to account for central extensions.

4.3 The proof in the generic case for positive central character

Convention 4.9. We restrict, as we may, to the specialization $\mathbb{H}_{n,m}$. If σ is a partition of n, recall from §3 that there is a real central character c_m^σ attached to it. The Hecke algebra $\mathbb{H}_{n,m}$ admits a family of discrete series $\mathsf{ds}_m(\sigma)$ with central character c_m^σ.

The starting point is the case $m \to \infty$ (so $m > n-1$), when the W_n-character is easy to understand.

We recall the formula for the generic degree (see [5], section 13.5) of the module of the finite Hecke algebra $\mathbb{H}_n^f(u,v)$ of type C_n with parameters u on the short roots and v on the long roots corresponding to $\gamma = \{(a_1,\dots,a_{k+1})(b_1,\dots,b_k)\} \in \widehat{W}_n$, in the bipartition notation. Here each partition is written in nondecreasing order, and assume that at least one of a_1 or b_1 are nonzero. Then, form the symbol $\begin{pmatrix} \lambda_1 & & \lambda_2 & \dots & \lambda_k & & \lambda_{k+1} \\ & \mu_1 & & \mu_2 & \dots & \mu_k & \end{pmatrix}$, where $\lambda_i = a_i + (i-1)$ and $\mu_j = b_j + (j-1)$. The generic degree $\mathrm{gd}(\gamma)$ with parameters u and v is

$$\frac{u^{m+\binom{m}{2}} v^{\sum_j b_j} P_n(u,v)(u-1)^n \prod_{i'<i}(u^{\lambda_i} - u^{\lambda_{i'}}) \prod_{j'<j}(u^{\mu_j} - u^{\mu_{j'}}) \prod_{i,j}(u^{\lambda_i - 1}v + u^{\mu_j})}{u^{\binom{2m-1}{2}+\binom{2m-3}{2}+\cdots} \left(\prod_i \prod_{l=1}^{\lambda_i}(u^l-1)(u^{l-1}v+1)\right)\left(\prod_j \prod_{l=1}^{\mu_j}(u^l-1)(u^{l+1}+v)\right)(u+v)^k}. \tag{4.20}$$

Proposition 4.10. *Assume that $m > n-1$. In $\mathbb{H}_{n,m}$, we have $C_{\mathsf{ds}_m(\sigma)} = \pm\frac{1}{2}$, for every partition σ of n.*

Proof. From [7] §4.7 we have

$$\mathsf{ds}_m(\sigma)|_{W_n} = \{\emptyset, {}^{\mathsf{t}}\sigma\}. \tag{4.21}$$

We identify the lowest power of q in the Euler expansion (4.9). This expansion becomes:

$$2(-1)^n[f_\pi, 1] = \frac{\mathrm{gd}\{\sigma,\emptyset\}}{P_n} + \sum_{i=1}^{n}(-1)^i \sum_{\gamma_1\boxtimes\gamma_2 \in \widehat{S_i\times S_{n-i}}} \frac{[\sigma : \gamma_1 \boxtimes \gamma_2]_{\widehat{S_i\times S_{n-i}}}\mathrm{gd}\{\gamma_2,\emptyset\}\mathrm{gd}\{\emptyset,{}^{\mathsf{t}}\gamma_1\}}{P_iP_{n-i}}. \tag{4.22}$$

We are interested, in the case when $m \to \infty$, to determine the lowest degree of q which may appear in the terms from (4.22). We use the generic degree formula (4.20) for $u = q$, $v = q^m$. The observation, using (4.20), is that for every $i \geq 1$, each factor of the form $\mathrm{gd}\{\emptyset, {}^{\mathsf{t}}\gamma_1\}$ in (4.22) contains a factor v^i and this dominates all factors in u. Therefore the lowest degree of q in every term for $i \geq 1$ is a linear nonconstant function in m. Moreover, in the first term in (4.22) (corresponding to $i = 0$), since the bipartition is $\{\sigma, \emptyset\}$, there is no factor of v present, and the lowest degree factor for $m \gg 0$ is a power of $u = q$ independent of m. This implies that the lowest power of q in the right hand side of (4.22) appears with coefficient one. On the other hand, in the product formula for $\mathrm{fd}(\pi)$ (see (4.2)), clearly the coefficient of the lowest power in a q-expansion is $\pm C_\pi$. □

Remark 4.11. Let $\sigma = (0 < a_1 \leq a_2 \leq \cdots \leq a_{l+1})$ denote a partition which we identify with the corresponding irreducible $\mathfrak{S}_n$-module. We define its lowest harmonic degree as $\mathrm{lhd}(\sigma) = \sum_{j=1}^{l}(l+1-j)a_j$. It is an elementary combinatorial calculation to see that the lowest powers of q in both (4.22) and (4.2) for $\mathsf{ds}_m(\sigma)$ are $q^{\mathrm{lhd}(\sigma)}$.

Theorem 4.12. *Assume that m is generic. Let σ be a partition of n. Then, in $\mathbb{H}_{n,m}$, we have $C_{\mathsf{ds}_m(\sigma)} = \pm\frac{1}{2}$.*

Proof. We analyze the behavior of $\mathrm{fd}(\mathsf{ds}_m(\sigma))$ as m crosses a critical value $m_0 \in \frac{1}{2}\mathbb{Z}$. Let h denote the number of balanced hooks at m_0 in σ. If π is any $\mathbb{H}_{n,m}$-module, recall that $\Theta_W(\pi)$ denotes the W_n-character of π.

Let $\mathsf{ds}_m^{\rightarrow}(\sigma), \mathsf{ds}_m^{\leftarrow}(\sigma)$ be the discrete series modules parameterized by σ for $m_0 - \frac{1}{2} < m < m_0$ and $m_0 < m < m_0 + \frac{1}{2}$, respectively. Then there exists a tempered delimit $\pi_m^{\rightarrow} \in \mathcal{D}_{m_0}(\sigma)$ such that

$$\Theta_W(\mathsf{ds}_m^{\rightarrow}(\sigma)) = \Theta_W(\pi_m^{\rightarrow}).$$

It is clear that also $[f_{\mathsf{ds}_m^{\rightarrow}(\sigma)}, 1] = [f_{\pi_m^{\rightarrow}}, 1]$, since $[f_\pi, 1]$ depends only on the W-character (for real positive central character). Now using Corollary 3.19 and Theorem 4.4 (2), we find that $[f_{\pi_m^{\rightarrow}}, 1] = \pm[f_{\mathsf{ds}_m^{\leftarrow}(\sigma)}, 1]$, and therefore $[f_{\mathsf{ds}_m^{\rightarrow}(\sigma)}, 1] = \pm[f_{\mathsf{ds}_m^{\leftarrow}(\sigma)}, 1]$, which implies the claim of Theorem 4.12. □

4.4 The proof in the generic case for nonpositive central character

Convention 4.13. As in the discussion around (4.10), let π, π_1, π_2 be discrete series of $\mathbb{H}_n$, $\mathbb{H}_{n,m_1}$, $\mathbb{H}_{n,m_2}$, respectively. We retain the notation from (4.10). In addition, we regard π_1 and π_2 as part of the families $\mathsf{ds}_{m_1}(\sigma_1)$ and $\mathsf{ds}_{m_2}(\sigma_2)$ for partitions σ_1 of k and σ_2 of $n-k$, respectively. Consequently, we denote the discrete series π of $\mathbb{H}_n$ by $\mathsf{ds}_{(m_+,m_-)}(\sigma_1, \sigma_2)$.

The proof is analogous to the case of positive real central character for $\mathbb{H}_{n,m}$. The analogous asymptotic region that we need is:

$$m_- \gg m_+ \gg 0: \quad m_1 = \frac{m_+ - m_-}{2} \to -\infty, \quad m_2 = \frac{m_+ + m_-}{2} \to +\infty. \tag{4.23}$$

Again by [7] §4.7, we have that

$$\mathsf{ds}_{m_1}(\sigma_1)|_{W_k} = \{\sigma_1, \emptyset\}, \quad \mathsf{ds}_{m_2}(\sigma_2)|_{W_{n-k}} = \{\emptyset, {}^{\mathsf{t}}\sigma_2\}. \tag{4.24}$$

From this and (4.10), we see that

$$\mathsf{ds}_{(m_+,m_-)}(\sigma_1, \sigma_2)|_{W_n} = \{\sigma_1, {}^{\mathsf{t}}\sigma_2\}. \tag{4.25}$$

In order to apply (4.9) we need to analyze the restrictions $\mathsf{ds}_{(m_+,m_-)}(\sigma_1, \sigma_2)|_{W_i \times W_{n-i}}$. The following lemma is sufficient for our purposes.

Lemma 4.14. *Assume that $m_- \gg m_+ \gg 0$. If $\mathrm{Hom}_{W_i \times W_{n-i}}(\gamma \boxtimes \delta, \mathsf{ds}_{(m_+,m_-)}(\sigma_1, \sigma_2)) \neq 0$ and $i \geq 1$, where $\gamma \in \widehat{W_i}$ and $\delta \in \widehat{W_{n-i}}$, then $\mathrm{Hom}_{W_1}(\mathsf{sgn}, \gamma) \neq 0$, where $W_1 \subset W_i$ denotes the reflection group generated by $s_0 := N_0|_{q=1}$.*

Proof. An algebraic family of modules of a finite group is rigid, and hence we have

$$\mathsf{ds}_{(m_+,m_-)}(\sigma_1, \sigma_2) \cong \lim_{q \to 1} \mathsf{ds}_{(m_+,m_-)}(\sigma_1, \sigma_2), \text{ as } W_i \times W_{n-i}\text{-modules},$$

for every $0 \leq i \leq n$.

Therefore, we replace $\mathsf{ds}_{m_1}(\sigma_1), \mathsf{ds}_{m_2}(\sigma_2), \mathsf{ds}_{(m_+,m_-)}(\sigma_1, \sigma_2)$ with their limits $q \to 1$ during this proof. We have presentations

$$\mathbb{C}[\widetilde{W_k}] \cong \mathbb{C}[W_k] \otimes \mathbb{C}[\epsilon_1^{\pm 1}, \ldots, \epsilon_k^{\pm 1}], \text{ and } \mathbb{C}[\widetilde{W}_{n-k}] \cong \mathbb{C}[W_{n-k}] \otimes \mathbb{C}[\epsilon_{k+1}^{\pm 1}, \ldots, \epsilon_n^{\pm 1}],$$

inside

$$\mathbb{C}[\widetilde{W_n}] \cong \mathbb{C}[W_n \ltimes X^*(T_n)] = \mathbb{C}[W_n] \otimes \mathbb{C}[\epsilon_1^{\pm 1}, \ldots, \epsilon_n^{\pm 1}].$$

By examining the central characters, we deduce that each $\epsilon_1, \dots, \epsilon_k$ acts on $\mathsf{ds}_{m_1}(\sigma_1) \subset \mathsf{ds}_{(m_+,m_-)}(\sigma_1, \sigma_2)$ by the uniform eigenvalue -1, while each $\epsilon_{k+1}, \dots, \epsilon_n$ acts on $\mathsf{ds}_{m_2}(\sigma_2) \subset \mathsf{ds}_{(m_+,m_-)}(\sigma_1, \sigma_2)$ by the uniform eigenvalue 1. Moreover, by (4.24), a long reflection of W_i acts on $\mathsf{ds}_{m_1}(\sigma_1)$ by the identity, while a long reflection of W_{n-i} acts on $\mathsf{ds}_{m_2}(\sigma_2)$ by the negative of the identity.

We have $s_0 = N_0|_{q=1} = \epsilon_1^{-1} \cdot s_\theta$, where s_θ is the long reflection of W_n corresponding to $\theta = 2\epsilon_1$. In particular, we have $s_0 \in \widetilde{W}_n$ and its W_n-conjugate act on

$$\mathsf{ds}_{m_1}(\sigma_1) \boxtimes \mathsf{ds}_{m_2}(\sigma_2) \subset \mathsf{ds}_{(m_+,m_-)}(\sigma_1, \sigma_2)$$

with uniform eigenvalue -1. Since

$$\mathbb{C}[W_n](\mathsf{ds}_{m_1}(\sigma_1) \boxtimes \mathsf{ds}_{m_2}(\sigma_2)) = \mathsf{ds}_{(m_+,m_-)}(\sigma_1, \sigma_2),$$

the same is true for $\mathsf{ds}_{(m_+,m_-)}(\sigma_1, \sigma_2)$.

The semisimplicity of the complex representations of finite groups implies then that s_0 acts on $\mathsf{ds}_{(m_+,m_-)}(\sigma_1, \sigma_2)$ by the negative of the identity.

Therefore, we conclude that the claim holds via the intermediate restriction $\langle s_0 \rangle \subset W_i \times W_{n-i} \subset \widetilde{W}_n$. $\square$

Proposition 4.15. *Assume that $m_- \gg m_+ \gg 0$. Fix $1 \leq k \leq n$ and partitions σ_1, σ_2 of k and $n-k$, respectively. Let $\pi := \mathsf{ds}_{(m_+,m_-)}(\sigma_1, \sigma_2)$ be as in Convention 4.13. Then we have $C_\pi = \pm\frac{1}{2}$.*

Proof. The proof is analogous to the proof of Proposition 4.10. In this case, the analogue of (4.22) is the formula:

$$2(-1)^n[f_\pi, 1] = \frac{\mathrm{gd}\{\sigma_2, {}^{\mathsf{t}}\sigma_1\}}{P_n(q, q^{m_+})} + \sum_{i=1}^{n}(-1)^i \sum_{\gamma \boxtimes \delta \in \widehat{W_i \times W_{n-i}}} \frac{[\pi : \gamma \boxtimes \delta]\mathrm{gd}(\gamma)\mathrm{gd}(\delta \otimes \mathsf{sgn})}{P_i(q, q^{m_-})P_{n-i}(q, q^{m_+})}. \quad (4.26)$$

In this formula, $\mathrm{gd}(\gamma)$ is computed in $\mathbb{H}_i^f(q, q^{m_-})$, while $\mathrm{gd}\{\sigma_2, {}^{\mathsf{t}}\sigma_1\}$ and $\mathrm{gd}(\delta)$ in $\mathbb{H}_n^f(q, q^{m_+})$ and $\mathbb{H}_{n-i}^f(q, q^{m_+})$, respectively. In light of Lemma 4.14, every $\gamma \in \widehat{W_i}$, $i \geq 1$ that appears in (4.26) contains the sign representation in its restriction to W_1. In other words, when written in the bipartition notation, $\gamma = \{\gamma_1, \gamma_2\}$, we have $\gamma_2 \neq \emptyset$. The formula for generic degree (4.20) implies then that $\mathrm{gd}(\gamma)$ contains the factor $v^{|\gamma_2|}$, where $v = q^{m_-}$, and $|\gamma_2|$ is the size of γ_2. Since $m_- \gg m_+ \gg 0$, the smallest power of q in the q-expansion is in the first term of (4.26), this being the only term of the sum that does not have as a factor a power of q^{m_-}. In particular, the coefficient of the lowest power of q in $[f_\pi, 1]$ in the region (4.23) is $\pm 1/2$. $\square$

Now we can prove Theorem 4.7 in the nonpositive case as well.

Proof of Theorem 4.7. Fix k such that $1 \leq k \leq n$, and fix σ_1, σ_2 partitions of k and $n-k$, respectively. Recall the family of discrete series $\mathsf{ds}_{(m_+,m_-)}(\sigma_1, \sigma_2)$ as in Convention 4.13. Proposition 4.15 verified the claim of Theorem 4.7 in the asymptotic region $m_- \gg m_+ \gg 0$.

To transfer the result from the asymptotic region to every generic region, one can proceed as in the proof of Theorem 4.12.

Fix a pair $(m_1^{-\infty}, m_2^{\infty})$ in the asymptotic region (4.23), and construct a line of parameters $t \mapsto (m_1^t, m_2^t)$, $t \geq 0$, where $m_1^t = m_1^{-\infty} + t$, $m_2^t = m_2^{\infty} - t$, a line of central characters $\mathsf{c}^t(\sigma_1, \sigma_2) := (-\mathsf{c}_{m_1^t}^{\sigma_1}, \mathsf{c}_{m_2^t}^{\sigma_2})$, and the corresponding lines of discrete series $\pi_1^t = \mathsf{ds}_{m_1^t}(\sigma_1)$, $\pi_2^t = \mathsf{ds}_{m_2^t}(\sigma_2)$, $\pi^t = \mathsf{ds}_{(m_+^t, m_-^t)}(\sigma_1, \sigma_2)$ of $\mathbb{H}_{k,m_1^t}$, $\mathbb{H}_{n-k,m_2^t}$, and $\mathbb{H}_n$, respectively. Recall

that $m_+^t = m_2^t + m_1^t$ (and hence $m_+^t = m_+^\infty$ is independent of t), and $m_-^t = m_2^t - m_1^t$. (Notice that these lines of parameters cover all possible values of the labels (m_+, m_-) as $(m_1^{-\infty}, m_2^\infty)$ varies in the asymptotic region.)

Let $t = t_0 > 0$ be a value for which the label $(m_+^{t_0}, m_-^{t_0})$ is critical, and let $U_\epsilon(t_0) := (t_0 - \epsilon, t_0 + \epsilon)$ be an interval such that the label (m_+^t, m_-^t) is generic for $t \in U_\epsilon(t_0) \setminus \{0\}$. By induction, we may assume that we know C_{π^t} for all $t \in U_\epsilon(t_0)$ with $t < t_0$. As in the proof of Theorem 4.12, we know that π_i^t, $\pi_i^{t'}$, $t_0 - \epsilon < t < t_0 < t' < t_0 + \epsilon$ are linked, $i = 1, 2$. The reduction to positive real central character implies that also π^t and $\pi^{t'}$ are linked. But then again Theorem 4.4 2) gives $C_{\pi^t} = \pm C_{\pi^{t'}}$. □

Remark 4.16. In order to apply these proofs to obtain the type D formulas in §4.2, in light of Proposition 3.34, it is sufficient to notice that if we have a W_n-type $\{\mu, \mu\}$ which splits $\{\mu, \mu\} = \{\mu, \mu\}_+ \oplus \{\mu, \mu\}_-$ as W_n^{D}-representations, then $\frac{1}{2} \lim_{m \to 0} \mathrm{gd}\{\mu, \mu\} = \mathrm{gd}^{\mathsf{D}}\{\mu, \mu\}_+ = \mathrm{gd}^{\mathsf{D}}\{\mu, \mu\}_-$ ([5], §13.5).

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